UNIVERSIDADE DE LISBOA
FACULDADE DE CIÊNCIAS
DEPARTAMENTO DE MATEMÁTICA

Bouncing solutions in a generalized Kepler problem

Alexandre Anahory de Sena Antunes Simões

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Carlota Rebelo

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Abstract

In [1], Rafael Ortega considered a periodically forced Kepler problem and proved the existence of linear motions with collisions. In this work, instead of the Kepler problem associated to the equation

$$\ddot{u} = -\frac{1}{u^2} + p(t)$$

we consider the problem

$$\ddot{u} = -\frac{1}{u^\alpha} + p(t), \quad \alpha \geqslant 2$$

and obtain results analogous to those in [1].

Resumo

Nesta dissertação estudamos um problema de Kepler generalizado. Este estudo inspira-se no artigo de Rafael Ortega [1] e tenta reproduzir os resultados obtidos, agora para uma equação mais geral, aumentando desta forma o ângulo de aplicação dos resultados.

O nosso interesse é estudar o comportamento das soluções da equação

$$\ddot{u} = -\frac{1}{u^\alpha} + p(t), \quad u > 0.$$

Historicamente o estudo das soluções da equação

$$\ddot{u} = -\frac{1}{u^2}, \quad u > 0$$

é conhecido como problema de Kepler. Esta é uma equação que aparece recorrentemente na Física, quer no modelo de gravitação Newtoniano, quer no modelo clássico de órbita de um electrão em torno do núcleo de um átomo. O problema que estudamos é uma generalização da equação do problema de Kepler para um expoente $\alpha \geqslant 2$ e com a adição de um termo periódico p.

No primeiro capítulo depois de algumas considerações preliminares sobre a equação estudamos algumas propriedades da solução clássica da equação. O primeiro resultado importante que obtemos é que a solução clássica, definida no intervalo maximal $]t_0, t_1[$ quando não pode ser prolongada para valores inferiores ao instante de tempo finito t_0 tem obrigatoriamente uma colisão neste ponto. Além disso, provamos adiante que a primeira derivada da solução em t_0 tende para $+\infty$. Entretanto, definimos o conceito de função de energia do sistema h e provamos que no instante t_0 esta é finita. Para isso calculamos as expansões assimptóticas da solução e da primeira derivada na vizinhança do instante t_0.

Seguidamente apresentamos uma proposição sobre o comportamento da solução no instante $t_1 < +\infty$. Na demonstração introduzimos a aplicação de inversão de tempo. Esta é uma técnica muito útil que nos permite concluir automaticamente para o instante t_1 todos os resultados anteriores que eram válidos para o instante t_0. Estes resultados são preparatórios e motivam as páginas seguintes. É nossa ambição provar a existência de uma única solução clássica da equação satisfazendo não um par de condições iniciais, mas um par de condições de colisão, isto é, dado um instante t_0 e um valor h_0 queremos provar a existência de uma única solução clássica com uma colisão em t_0 e cujo valor da função de energia é h_0 no instante t_0.

Por fim, apresentamos a técnica de regularização da equação, através da definição do integral de Sundman. O campo vectorial associado ao sistema regularizado está definido em todo o espaço euclideano o que nos vai permitir transformar o problema com condições de colisão num problema de Cauchy para o sistema regularizado.

No capítulo 2 introduzimos o conceito de solução de colisão. A solução de colisão destingue-se da solução clássica por estar definida em todo $\mathbb{R}$ e por admitir que a solução atinja o valor $u = 0$ num conjunto contável de instantes. A solução de colisão tem de satisfazer a equação em todos os intervalos abertos onde seja diferente de zero.

O objectivo deste capítulo é provar a existência de uma única solução de colisão satisfazendo uma condição de colisão. Começa com um resultado de dependência contínua que afirma que a solução da equação com condições iniciais $u(t_0) = \varepsilon$ e função energia com valor h_0 converge uniformemente para a solução da equação com condições de colisão (t_0, h_0) quando ε tende para zero.

Segue-se o subcapítulo de comparação de soluções. Dadas duas soluções clássicas da equação u_1 e u_2 e sabendo que num dado instante τ

$$u_1(\tau) \leqslant u_2(\tau), \quad \dot{u}_1(\tau) \leqslant \dot{u}_2(\tau),$$

esta ordem mantém-se para instantes $t > \tau$. O mesmo se verifica comparando as soluções das equações

$$\ddot{u} = -\frac{1}{u^\alpha} + p_i(t)$$

com $i = 1, 2$ para as quais temos as mesmas desigualdades no instante τ e $p_1 \leqslant p_2$. No caso de soluções com condições de colisão $h_{01} \leqslant h_{02}$ onde h_{0i} é a função de energia da solução u_i no instante t_0, obtemos o mesmo resultado.

De seguida, expomos um pequeno capítulo sobre a equação autónoma obtida supondo p constante. Definimos alguns dos conceitos clássicos da mecânica como as energias mecânica, cinética e potencial do sistema. A partir destes conceitos é possível fazer uma análise da dinâmica do sistema. Para concluir apresentamos alguns resultados sobre o comprimento do intervalo maximal de definição da solução.

O capítulo finaliza com a demonstração que existe uma única solução de colisão. Esta solução é construída colando as soluções clássicas pelos extremos. Dadas as condições de colisão (t_0, h_0) consideramos primeiro a solução clássica que as satisfaz. Então tomando, caso exista, o instante de colisão seguinte a que chamamos t_1, extraímos as condições de colisão (t_1, h_1). Por fim, juntamos as soluções clássicas que correspondem as estes dois pares de condições de colisão. Repetindo este processo recursivamente obtemos uma solução de colisão. A demonstração ocupa-se essencialmente com garantir que podemos realizar esta operação.

No terceiro capítulo definimos a aplicação do sucessor $\mathcal{P}$. Esta aplicação leva um par de condições de colisão (t_0, h_0) para o par (t_1, h_1) onde t_1 é o instante de colisão seguinte e h_1 o valor da função de energia neste instante. Esta aplicação só faz sentido se o instante t_1 existir. Neste caso, note-se que a aplicação está bem definida porque a solução de colisão é única. Sendo assim o domínio de $\mathcal{P}$ é o conjunto

$$D = \{(t_0, h_0) \in \mathbb{R}^2 : \ t_1 < +\infty\}.$$

Na primeira parte do terceiro capítulo estudamos algumas propriedades de $\mathcal{P}$ e do conjunto D, descritas na Proposição 3.6. Para a demonstrar precisamos de provar vários resultados preliminares. Entre eles destacamos o Lemma 3.5, que nos equipa com condições suficientes para garantir que o instante de colisão t_1 é finito.

A secção seguinte é tecnicamente a mais complicada. Na Secção 3.2 pretendemos provar que a aplicação $\mathcal{P}$ é simpléctica exacta. Começamos por discutir a diferenciabilidade da aplicação $\mathcal{P}$. No resultado definimos as aplicações τ e $\mathcal{H}$. A aplicação τ é uma generalização de uma aplicação diferenciável que no ponto (t_0, h_0) tenha imagem t_1; $\mathcal{H}$ é uma generalização de uma aplicação diferenciável que no ponto (t_0, h_0) tenha imagem h_1.

Na última parte do Capítulo 3, expomos a demonstração de que $\mathcal{P}$ é simpléctica exacta, na qual o resultado anterior é fundamental. Por definição, temos que provar que a forma diferencial

$$h_1 dt_1 - h_0 dt_0$$

é exacta. Para isso recorremos a um resultado de formas diferenciáveis que nos garante que se esta forma for fechada então é exacta, o que reduz a demonstração a um exercício de integrais de linha.

Por fim, no último capítulo aplicamos o Teorema de Poincaré-Birkhoff para concluir que existem pelo menos duas soluções periódicas. Aqui já quase todas as hipóteses do teorema estão satisfeitas. Todo o trabalho que resta é verificar a última das condições exigidas pelo teorema.

O teorema de Poincaré-Birkhoff tem uma história centenária e conturbada, feita de demonstrações incompletas, demonstrações imperceptíveis e tentativas de aperfeiçoamento mal sucedidas. No entanto, nos últimos anos têm sido alcançados alguns avanços na sua aplicabilidade. Nesta dissertação usamos um enunciado simplificado que pode ser consultado em [1], onde vem demonstrado. Este teorema é uma ferramenta clássica que garante a existência de pelo menos dois pontos fixos de uma aplicação.

Chamamos a atenção que no apêndice, além de resultados auxiliares que utilizamos ao longo da dissertação, consta também uma secção onde explicamos a notação usada ao longo do trabalho, apontamos as noções básicas de equações diferenciais e alguns dos resultados clássicos que necessitamos. O leitor menos experimentado na teoria clássica de equações diferenciais deverá começar por entender estes resultados.

Contents

Introduction

In this work we study a generalized Kepler problem. We were inspired by Rafael Ortega in [1] where the author considered the equation

$$\ddot{u} = -\frac{1}{u^2} + p(t), \quad u > 0,$$

and we try to obtain analogous results to those of Ortega, but now for a wider range of equations. The existence of periodic solutions for this equation has been studied by different authors in the past recent years. We mention [3], where the authors give a necessary and sufficient condition for the existence of a periodic solution, and [4], where the results obtained imply uniqueness of the periodic solution and that it is unstable. Rafael Ortega in his article considers solutions with collisions and it turns out that the dynamics happens to be very rich provided we give a proper definition - the bouncing solution.

Our main interest is to study the behaviour of the solutions of the equation

$$\ddot{u} = -\frac{1}{u^\alpha} + p(t), \quad u > 0, \quad \alpha \geqslant 2.$$

Historically, the study of the solutions of equation

$$\ddot{u} = -\frac{1}{u^2}, \quad u > 0$$

is known as Kepler problem. This equation comes up rather frequently on Physics. It governs the gravitational force on Newton's model as well as it describes the motion of the electron orbiting its nucleus on the classical model of the atom. So we may say that we study a generalized Kepler problem for an exponent $\alpha \geqslant 2$ to which we add a periodic force p.

In the first chapter, after some preliminary results about the equation we study the properties of the classical solutions. The first main result obtained states that the classical solution defined on the maximal interval of definition $]t_0, t_1[$ has a collision at the instant t_0 whenever t_0 has a finite value. Also we prove that the first derivative of the solution tends to $+\infty$ near this instant t_0.

Next we present a proposition that describes the behaviour of the solution at the instant $t_1 < +\infty$. In its proof we introduce the time reversing map, which is a very useful tool, that immediately enables us to apply to t_1 all the former results we knew for t_0. All these results are nothing but warm up for what is coming next. It is our goal to prove the existence of one and only one classical solution satisfying a pair of collision conditions instead of the usual initial conditions, i.e., given two numbers t_0, h_0 we want to prove the existence of a unique solution with a collision at the instant t_0 and such that the energy function is h_0 at the same instant.

To finish the chapter, we make use of a technique: the regularization of the equation. By defining the Sundman integral we obtain a regularized system which is a system linked to our equation and moreover it is very well behaved, i.e., the vector field associated to this regularized system is defined in the whole euclidean space, containing no singular points. In this way we reduce the problem with collision conditions to a Cauchy problem for the regularized system.

In the second chapter we introduce the concept of bouncing solution. It is different from the classical solution by the fact that it is defined in $\mathbb{R}$ and the value $u = 0$ is allowed as long as it is reached only in a countable set. In all open intervals where the bouncing solution is not zero it must satisfy the equation.

In this chapter we prove the existence of a unique bouncing solution satisfying some given collision conditions. First we present a result on continuous dependence which states that the solution satisfying the initial conditions $u(t_0) = \varepsilon$ and h_0 and such that the value of the energy function at the instant t_0 is h_0 converges uniformly to the solution satisfying the collision conditions (t_0, h_0) when we let ε go to zero.

Next you find a section on comparison of solutions. Given two classical solutions u_1 and u_2, if in some instant τ

$$u_1(\tau) \leqslant u_2(\tau), \quad \dot{u}_1(\tau) \leqslant \dot{u}_2(\tau),$$

then this order holds for instants $t > \tau$. The same happens if we compare the solutions of the equations

$$\ddot{u} = -\frac{1}{u^\alpha} + p_i(t),$$

with $i = 1, 2$, for which we know the same inequalities at instant τ and we have $p_1 \leqslant p_2$. When we have collision conditions with $h_{01} \leqslant h_{02}$ where h_{0i} is the value of the energy function at the instant t_0 we also obtain the same result.

Then we have a chapter on the autonomous equation obtained supposing that the function p is constant. We define some basic concepts from classical mechanics. From this we can make an analysis of the dynamics. To conclude we present some useful results about the maximal interval of definition of the solutions.

Last but not the least we prove the existence of only one bouncing solution. Bouncing solutions are obtained by gluing together classical solutions in an appropriate fashion. Given some collision conditions (t_0, h_0) we take the classical solution satisfying these conditions. Then we look at the next collision and extract another pair of collision conditions (t_1, h_1). Then we glue together the classical solutions corresponding to these two pairs of collision conditions. And we repeat this process recursively until we obtain a bouncing solution.

In the third chapter we define the successor map $\mathcal{P}$. This map assigns to each pair of collision conditions (t_0, h_0) another pair of collision conditions (t_1, h_1) where t_1 is the next instant of collision. This map is well defined in the domain

$$D = \{(t_0, h_0) \in \mathbb{R}^2 : \; t_1 < +\infty\},$$

because the bouncing solution satisfying the collision conditions (t_0, h_0) is unique.

We start with the study of some properties of the successor map as well as those of the set D, which are stated in Proposition 3.6. To prove it we need several preliminary results. We highlight Lemma 3.5., which gives us sufficient conditions in order to guarantee that the instant t_1 is finite.

Next section is technically the most difficult. In Section 3.1. we intend to prove that the successor map is exact symplectic (the definition is in the appendix). As soon as we accomplish this we will be able to apply the Poincaré-Birkhoff theorem to the successor map. So, first we discuss the differentiability of $\mathcal{P}$. At this point we define the maps τ and $\mathcal{H}$. We can see the first map as a generalization of the differential map that sends each pair $(t_0, h_0) \in D$ to t_1. Analogously $\mathcal{H}$ can be seen as a generalization of a map that sends (t_0, h_0) to h_1, i.e., the value of the energy function at the instant t_1.

The last part of Chapter 3 is dedicated to prove that $\mathcal{P}$ is indeed exact symplectic. By definition we must prove that the differential form

$$h_1 dt_1 - h_0 dt_0$$

is exact. For that reason we make use of a theorem stating that if this form is closed then it is exact. Thanks to this we reduce the proof into an exercise of line integrals.

In the final chapter we apply the Poincaré-Birkhoff theorem to conclude there exist at least two periodic bouncing solutions. At this stage almost all conditions of the theorem are met. The hard work is just to prove one last condition.

1 Preliminary results

1.1 The equation and the classical solution

Let us consider the differential equation

$$\ddot{u} = -\frac{1}{u^\alpha} + p(t), \quad u > 0 \tag{1}$$

where $p : \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous and 2π-periodic function and $\alpha \geqslant 2$. We want to study the existence of generalized periodic solutions and their properties. By generalized periodic solution we mean a periodic solution which can attain the value zero for some values of the domain (see the definition of bouncing solution in Chapter 2). The values of t for which $u(t) = 0$ will be called *collision* instants.

By now let $p : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous and bounded function with $\|p\|_\infty := \sup_{t\in\mathbb{R}} |p(t)|$. Equation (1) can easily be rewritten as a first order system

$$\begin{cases} \dot{u} = v \\ \dot{v} = -\frac{1}{u^\alpha} + p(t). \end{cases}$$

If we define $x(t) = (u(t), v(t))$ we get the system

$$\dot{x} = X(t, x)$$

where $X(t, x) = \left(v, -\frac{1}{u^\alpha} + p(t)\right)$ is the vector field associated to the differential equation. The vector field $X : \mathbb{R}\times]0, +\infty[\times\mathbb{R} \longrightarrow \mathbb{R}^2$ is a continuous function from its domain $D := \mathbb{R}\times]0, +\infty[\times\mathbb{R}$ to $\mathbb{R}^2$. Moreover D is an open and connected set and the Jacobian matrix

$$\frac{\partial X}{\partial x}(t, x) = \begin{pmatrix} 0 & 1 \\ \frac{\alpha}{u^{\alpha+1}} & 0 \end{pmatrix}$$

exists for all $(t, x) \in D$ and is continuous in D. Therefore using standard results from the classical theory of differential equations we know that there is a unique classical solution for each initial condition $(t^*, x^*) \in D$ and also that there exists a maximal solution for the Cauchy problem.

Now that we know there is a maximal solution for this problem let us denote by $]t_0, t_1[$ the maximal interval where the solution is defined and suppose that $t_0 > -\infty$. By Theorem 5.12. we have that one of the following holds:

(*i*) $\lim_{t\to t_0^+} \|x(t)\| = +\infty$.

(*ii*) There exists a decreasing sequence $(t_n)_{n\in\mathbb{N}}$ converging to t_0 such that $x(t_n) \to \xi$ where $(t_0, \xi) \in \partial D$, i.e., (t_0, ξ) is in the boundary of D.

Notice that condition (*ii*) implies that $u(t_n) \to 0$ which again implies

$$\liminf_{t\to t_0^+} u(t) = 0.$$

Proposition 1.1. *Let u be a solution of (1) defined after an instant $t_0 > -\infty$.*
Then $\liminf_{t\to t_0{}^+} u(t) = 0$ *always holds.*

Proof. Let us argue by contradiction. Suppose it was indeed possible that $\liminf_{t\to t_0{}^+} u(t) > 0$. Then there would exist $\delta, \rho > 0$ such that $u(t) \geqslant \delta$, for $t \in]t_0, t_0 + \rho[:= I_\rho$.

Then $|\ddot{u}| \leqslant |\frac{1}{u^\alpha}| + |p(t)| < \frac{1}{\delta^\alpha} + \|p\|_\infty := M_1$ in I_ρ.

Then for every t_1, $t_2 \in I_\rho$ it follows

(a)

$$|\dot{u}(t_1) - \dot{u}(t_2)| = |\int_{t_2}^{t_1} \ddot{u}(t)\ dt| \leqslant \int_{t_2}^{t_1} |\ddot{u}(t)|\ dt < \rho M_1.$$

Fixing $a \in I_\rho$, then for every $t \in I_\rho$

$$|\dot{u}(t)| \leqslant |\dot{u}(t) - \dot{u}(a)| + |\dot{u}(a)| < \rho M_1 + |\dot{u}(a)| := M_2.$$

Then for every t_1, $t_2 \in I_\rho$ it follows

(b)

$$|u(t_1) - u(t_2)| = |\int_{t_2}^{t_1} \dot{u}(t)\ dt| \leqslant \int_{t_2}^{t_1} |\dot{u}(t)|\ dt < \rho M_2.$$

Proceeding analogously we find that for every $t \in I_\rho$

$$|u(t)| < \rho M_2 + |u(a)|.$$

Therefore $\lim\limits_{t\to t_0{}^+} \|x(t)\| < +\infty$ and neither of statements (i) and (ii) hold. This is a contradiction.
□

The next result shows that not only the inferior limit is zero in a neighbourhood of the collision point as well as the limit itself is zero.

Proposition 1.2. *Let u be a solution of (1) defined after an instant $t_0 > -\infty$.*
Then $\lim\limits_{t\to t_0{}^+} u(t) = 0$, i.e., t_0 is a collision point.

Proof. Let us again argue by contradiction. Suppose there exists $M > 0$ such that

$$\limsup_{t\to t_0{}^+} u(t) > M.$$

Then we can find points t_1, t_2, t_3 so close to t_0 as we wish such that

$$t_0 < t_3 < t_2 < t_1,$$

$$\begin{cases} u(t_1) := u_1 = \frac{M}{10} \\ u(t_2) := u_2 = M \\ u(t_3) := u_3 = \frac{M}{10} \end{cases}$$

and also

$$u(t) \geqslant \frac{M}{10} \quad \forall t \in [t_3, t_1].$$

This implies that $|\ddot{u}(t)| \leqslant |\frac{1}{u^\alpha}| + |p(t)| \leqslant \left(\frac{10}{M}\right)^\alpha + \|p\|_\infty := K_1$ for any $t \in [t_3, t_1]$.

Let us choose t_1 such that

$$|t_1 - t_0|^2 < \frac{M}{10K_1}$$

and compute the integral

$$\int_{t_3}^{t} (t-s)\ddot{u}(s)\ ds = \dot{u}(t_3)(t_3 - t) - u_3 + u(t).$$

Denoting $\dot{u}(t_3) = \dot{u}_3$ and recalling that $u_3 = \frac{M}{10}$ we get

$$u(t) = \frac{M}{10} + \dot{u}_3(t - t_3) + \int_{t_3}^{t} (t-s)\ddot{u}(s)\ ds.$$

On the interval $[t_3, t_1]$, $\ddot{u}(s) \leqslant K_1$. Therefore,

$$u(t) \leqslant \frac{M}{10} + \dot{u}_3(t - t_3) + K_1 \int_{t_3}^{t} (t-s)\ ds$$

on this interval. In particular, setting $t = t_2$ with $u(t_2) = M$ and noting that $t_2 - s \leqslant t_2 - t_3$ when $s \in [t_3, t_2]$ we can conclude

$$\dot{u}_3 \geqslant \frac{9M}{10}(t_2 - t_3)^{-1} - K_1(t_2 - t_3).$$

Now recall the way how t_1 was chosen

$$(t_2 - t_3)^2 < (t_1 - t_0)^2 < \frac{M}{10K_1}.$$

This implies

$$K_1 < \frac{M}{10}(t_2 - t_3)^{-2}$$

and hence

$$\dot{u}_3 > \frac{8M}{10}(t_2 - t_3)^{-1}.$$

Finally, let us get the contradiction. If we set $t = t_1$ then

$$u_1 = \frac{M}{10} + \dot{u}_3(t_1 - t_3) + \int_{t_3}^{t_1} (t_1 - s)\ddot{u}(s)\ ds.$$

Observe that

$$\begin{aligned} u_1 &\geqslant \dot{u}_3(t_1 - t_3) - \frac{M}{10} - \left|\int_{t_3}^{t_1} (t_1 - s)\ddot{u}(s)\ ds\right| \\ &\geqslant \dot{u}_3(t_1 - t_3) - \frac{M}{10} - K_1(t_1 - t_3)^2 \\ &> \frac{8M}{10}\frac{(t_1 - t_3)}{(t_2 - t_3)} - \frac{M}{10} - K_1(t_1 - t_3)^2 \end{aligned}$$

As previously $K_1 < \frac{M}{10}(t_1 - t_3)^{-2}$. Therefore

$$u_1 > \frac{8M}{10}\frac{(t_1 - t_3)}{(t_2 - t_3)} - \frac{2M}{10}.$$

Moreover, $t_1 - t_3 > t_2 - t_3$ so

$$u_1 > \frac{8M}{10} - \frac{2M}{10} = \frac{6M}{10}.$$

However $u_1 = \frac{M}{10}$. This is a contradiction. Then $\limsup_{t\to t_0^+} u(t) = 0$ and $\lim\limits_{t\to t_0^+} u(t) = 0$ follows. $\square$

From now on, $]t_0, t_1[$ will always denote the maximal interval of definition of the solution u of equation (1).

Next we define the concept of energy function. We will prove that it is finite when the solution approaches the instant of collision t_0.

Definition 1.1. We call *energy function* the map

$$h :]0, +\infty[\times] - \infty, +\infty[\longrightarrow \mathbb{R}$$

defined by

$$h(u, v) = \frac{1}{2}v^2 - \frac{1}{\alpha - 1}\frac{1}{u^{\alpha-1}}.$$

Given $u(t)$ a solution of equation (1), we denote $h(t) := h(u(t), \dot{u}(t))$, and we say that h is the *energy function of the system*.

Proposition 1.3. *Let u be a solution of (1) and $t_0 > -\infty$.*

Then the energy function of the system is finite as the solution approaches a collision point, i.e., $\lim\limits_{t\to t_0^+} h(t) := h_0$ exists.

To prove this we will need first to compute some asymptotic expansions for $u(t)$ and $\dot{u}(t)$ in a neighbourhood of t_0. We can accomplish this as soon as we establish the following lemmas.

Lemma 1.4. *Let u be a solution of (1) and $t_0 > -\infty$. Then $\lim\limits_{t\to t_0^+} u^{\frac{\alpha}{2}}\dot{u} = 0$*

Proof. First we prove that $\dot{u}$ maintains the sign in a neighbourhood of t_0. Note that

$$\forall \delta > 0, \quad \exists\, t^* \in]t_0, t_0 + \delta[: \quad \dot{u}(t^*) > 0.$$

If this was not the case then there would exist $\delta > 0$ such that $\dot{u}(t) \leqslant 0$, $\forall t \in]t_0, t_0 + \delta[$ and hence $u(t_0) \geqslant u(t_0 + \delta) \neq 0$, which is a contradiction because $u(t)$ is positive in the maximal interval $]t_0, t_1[$.

Moreover $\exists\, \varepsilon > 0$ such that $\ddot{u}(t) \leqslant 0$, $\forall t \in]t_0, t_0 + \varepsilon[$.

Let us choose $\delta = \varepsilon$. Therefore $\exists\ t^* \in]t_0, t_0 + \varepsilon[$ such that $\dot{u}(t^*) > 0$.

Then

$$\dot{u}(t^*) - \dot{u}(t) = \int_t^{t^*} \ddot{u}(s)\ ds < 0, \quad \forall t \in]t_0, t^*[$$

and so

$$\dot{u}(t) > \dot{u}(t^*) > 0, \quad \forall t \in]t_0, t^*[.$$

Therefore there is a neighbourhood of t_0 in which $\dot{u}$ is positive.

Now, multiplying equation (1) by $\dot{u}(t)$ we obtain

$$\dot{u}\ddot{u} = -\frac{\dot{u}}{u^\alpha} + \dot{u}p,$$

that is

$$\frac{d}{dt}\left[\frac{1}{2}\dot{u}^2\right] = -\frac{\dot{u}}{u^\alpha} + \dot{u}p.$$

Integrating between $t \in]t_0, t^*[$ and t^* we find that

$$\frac{1}{2}\dot{u}^2(t^*) - \frac{1}{2}\dot{u}^2(t) = -\left[\frac{1}{1-\alpha}\frac{1}{u^{\alpha-1}}\right]_t^{t^*} + \int_t^{t^*} \dot{u}(s)p(s)\ ds.$$

To the last integral we can apply the mean value theorem and conclude that

$$\frac{1}{2}\dot{u}^2(t^*) - \frac{1}{2}\dot{u}^2(t) = -\left[\frac{1}{1-\alpha}\frac{1}{u^{\alpha-1}}\right]_t^{t^*} + \ p(\xi)\int_t^{t^*} \dot{u}(s)\ ds, \quad \text{for some } \xi \in [t, t^*],$$

and so, noting that the variable ξ depends on t, we find

$$\dot{u}^2(t) = \frac{2}{1-\alpha}\frac{1}{u^{\alpha-1}(t^*)} - \frac{2}{1-\alpha}\frac{1}{u^{\alpha-1}(t)} - 2p\left(\xi(t)\right)\left(u(t^*) - u(t)\right) + \dot{u}^2(t^*).$$

So in fact we have

$$\dot{u}^2(t) = -\frac{2}{1-\alpha}\frac{1}{u^{\alpha-1}(t)} + 2p\left(\xi(t)\right)u(t) - 2p\left(\xi(t)\right)u(t^*) + A, \tag{2}$$

where A is a constant depending just on t^*.

Therefore,

$$u^\alpha(t)\dot{u}^2(t) = -\frac{2}{1-\alpha}u(t) + 2p\left(\xi(t)\right)u^{\alpha+1}(t) - 2p\left(\xi(t)\right)u(t^*)u^\alpha(t) + Au^\alpha(t).$$

As $\lim_{t\to t_0} u(t) = 0$ and p is bounded it follows

$$\lim_{t\to t_0} u^\alpha(t)\dot{u}^2(t) = 0.$$

Finally, taking square roots we end up with

$$\lim_{t\to t_0} u^{\frac{\alpha}{2}}(t)\dot{u}(t) = 0.$$

□

As consequence of the previous lemma we can prove that the derivative approaches infinity as t approaches t_0.

Lemma 1.5. $\lim_{t\to t_0^+} \dot{u}(t) = +\infty.$

Proof. In the previous lemma we have seen that there is a neighbourhood of t_0, which we have designated by $]t_0, t^*[$ in which $\dot{u}(t) > 0$.

Moreover by (2) we deduce that

$$\dot{u}(t) \longrightarrow \pm\infty$$

as $t \to t_0^+$. Therefore it must be true that

$$\lim_{t\to t_0^+} \dot{u}(t) = +\infty.$$

$\square$

Lemma 1.6. *Let u be a solution of (1) and $t_0 > -\infty$. Then the energy function of the system h is bounded in a neighbourhood of t_0.*

Proof. Recall

$$h(t) = \frac{1}{2}\dot{u}^2(t) - \frac{1}{\alpha - 1}\frac{1}{u^{\alpha-1}(t)}$$

is the energy function. We have that

$$\dot{h}(t) = \dot{u}(t)p(t)$$

is its derivative.

Let us choose as before a point t^* in a neighbourhood of t_0, say $]t_0, t_0 + \varepsilon[$, in which $\dot{u}$ is positive. Integrating the derivative of the energy between some $t \in]t_0, t^*[$ and t^* and then applying the mean value theorem we conclude

$$h(t^*) - h(t) = \int_t^{t^*} \dot{u}(s)p(s)\ ds = p(\xi)(u(t^*) - u(t)), \quad \xi \in [t, t^*].$$

As p is a bounded function and u stays bounded as t approaches t_0 we conclude that $h(t)$ remains bounded as t approaches t_0.

$\square$

Now we are ready to compute the asymptotic expansions of u and $\dot{u}$, where u is a solution of the equation (1).

Proposition 1.7. *Given u a solution of equation defined in $]t_0, t_1[$ we have the following expansions near t_0:*

1.

$$u(t) = \left[\frac{\alpha+1}{2\alpha}\left(\frac{2\alpha^2}{\alpha-1}\right)^{\frac{1}{2}}\right]^{\frac{2}{\alpha+1}} (t-t_0)^{\frac{2}{\alpha+1}} + O((t-t_0)^{\frac{4}{\alpha+1}}).$$

2.
$$\dot{u}(t) = \frac{2}{\alpha+1}\left(\frac{\alpha^2+2\alpha+1}{2(\alpha-1)}\right)^{\frac{1}{\alpha+1}}(t-t_0)^{\frac{1-\alpha}{\alpha+1}} + O\left((t-t_0)^{\frac{3-\alpha}{\alpha+1}}\right).$$

Proof. First note that

$$\frac{d^2}{dt^2}(u^\alpha) = \alpha(\alpha-1)u^{\alpha-2}\dot{u}^2 + \alpha u^{\alpha-1}\ddot{u}. \tag{3}$$

We drop the argument on u and its derivatives for simplicity. We know that

$$\dot{u}^2 = 2h(t) + \frac{2}{(\alpha-1)u^{\alpha-1}} \tag{4}$$

and that

$$\ddot{u} = -\frac{1}{u^\alpha} + p(t). \tag{5}$$

Inserting (4) and (5) in (3) we get

$$\begin{aligned}\frac{d^2}{dt^2}(u^\alpha) &= \alpha(\alpha-1)u^{\alpha-2}\left(2h(t) + \frac{2}{(\alpha-1)u^{\alpha-1}}\right) + \alpha u^{\alpha-1}\left(-\frac{1}{u^\alpha} + p(t)\right)\\ &= 2\alpha(\alpha-1)u^{\alpha-2}h(t) + \frac{\alpha}{u} + \alpha u^{\alpha-1}p(t)\end{aligned}$$

Let us define $R = u^\alpha$ and $b(t) = 2\alpha(\alpha-1)u^{\alpha-2}h(t) + \alpha u^{\alpha-1}p(t)$. Note that b is a bounded function in a neighbourhood of t_0.

Then we find that
$$\ddot{R} = \frac{\alpha}{R^{\frac{1}{\alpha}}} + b(t).$$

Multiplying this equation by $\dot{R}$, observing that $\dot{R}\ddot{R} = \frac{d}{dt}\left(\frac{1}{2}\dot{R}^2\right)$ and integrating we get

$$\int_{t_0}^{t}\frac{d}{ds}\left(\frac{1}{2}\dot{R}^2\right)\,ds = \int_{t_0}^{t}\alpha\frac{\dot{R}}{R^{\frac{1}{\alpha}}}\,ds + \int_{t_0}^{t}b(s)\dot{R}\,ds.$$

We have to be careful with the integration, for the function u is not defined at t_0. However, as
$$\lim_{t\to t_0} R(t) = 0$$
and
$$\lim_{t\to t_0}\dot{R}(t) = \lim_{t\to t_0}\alpha u^{\alpha-1}\dot{u}(t) = \lim_{t\to t_0}\alpha u^{\frac{\alpha}{2}-1}u^{\frac{\alpha}{2}}\dot{u}(t) = 0,$$
by Lemma 1.4, and because $\frac{\alpha}{2}-1 \geqslant 0$, then it follows that

$$\dot{R}^2(t) = \frac{2\alpha^2}{\alpha-1}R^{\frac{\alpha-1}{\alpha}}(t) + 2\int_{t_0}^{t}b(s)\dot{R}(s)\,ds.$$

To the second integral, as previously we apply the mean value theorem to conclude that there exists $\xi(t) \in [t_0, t]$ such that

$$\int_{t_0}^{t} b(s)\dot{R}(s)\ ds = b(\xi(t)) \int_{t_0}^{t} \dot{R}(s)\ ds$$

Therefore, defining $b(t) = b(\xi(t))$ we obtain

$$\dot{R} = \left(\frac{2\alpha^2}{\alpha - 1} R^{\frac{\alpha-1}{\alpha}} + 2b(t)R(t) \right)^{\frac{1}{2}}.$$

Again let us operate another change of variables. This time let

$$z = R^{\frac{\alpha+1}{2\alpha}}.$$

Then, $R = z^{\frac{2\alpha}{\alpha+1}}$. Therefore, putting this information into the equation for $\dot{R}$ we get

$$\dot{z} = \frac{\alpha+1}{2\alpha} \left[\frac{2\alpha^2}{\alpha - 1} + 2b(t)z^{\frac{2}{\alpha+1}} \right]^{\frac{1}{2}}.$$

Integrating between t_0 and t and using the fact that $\lim_{t\to t_0} z(t) = 0$ it follows

$$z(t) = \frac{\alpha+1}{2\alpha} \int_{t_0}^{t} \left[\frac{2\alpha^2}{\alpha - 1} + 2b(s)z^{\frac{2}{\alpha+1}} \right]^{\frac{1}{2}}\ ds.$$

Notice that

$$\left| \frac{z(t)}{t - t_0} \right| = \left| \frac{z(t) - z(t_0)}{t - t_0} \right| = |\dot{z}(\xi)|, \text{ with } \xi \in]t_0, t[.$$

Furthermore, $|\dot{z}(\xi)|$ is bounded in a neighbourhood of t_0 as the function b is also bounded and as z converges to 0 as t approaches t_0. Therefore we conclude that $z(t) = O(t - t_0)$.

Going back to the formula for z and substituting this new information we find

$$z(t) = \frac{\alpha+1}{2\alpha} \int_{t_0}^{t} \left[\frac{2\alpha^2}{\alpha - 1} + 2b(s)O(s - t_0)^{\frac{2}{\alpha+1}} \right]^{\frac{1}{2}}\ ds.$$

It is easy to verify that $(O(s - t_0))^{\frac{2}{\alpha+1}} \subseteq O((s - t_0)^{\frac{2}{\alpha+1}})$ and that

$$2b(s)O((s - t_0)^{\frac{2}{\alpha+1}}) \subseteq O((s - t_0)^{\frac{2}{\alpha+1}}),$$

noting that b is a bounded function.

Then

$$z(t) = \frac{\alpha+1}{2\alpha} \int_{t_0}^{t} \left[\frac{2\alpha^2}{\alpha - 1} + O((s - t_0)^{\frac{2}{\alpha+1}}) \right]^{\frac{1}{2}}\ ds.$$

Using Proposition 5.2. on the integrand we deduce

$$z(t) = \frac{\alpha+1}{2\alpha} \int_{t_0}^{t} \left(\frac{2\alpha^2}{\alpha-1}\right)^{\frac{1}{2}} + O((s-t_0)^{\frac{2}{\alpha+1}})\ ds.$$

As $\int_{t_0}^{t} O((s-t_0)^{\frac{2}{\alpha+1}})\ ds = (t-t_0)O((t-t_0)^{\frac{2}{\alpha+1}})$, also by Proposition 5.2., it follows

$$z(t) = \frac{\alpha+1}{2\alpha} \left(\frac{2\alpha^2}{\alpha-1}\right)^{\frac{1}{2}} (t-t_0) \left[1 + O((t-t_0)^{\frac{2}{\alpha+1}})\right].$$

Finally, as $u = z^{\frac{2}{\alpha+1}}$ we get

$$u(t) = \left[\frac{\alpha+1}{2\alpha} \left(\frac{2\alpha^2}{\alpha-1}\right)^{\frac{1}{2}}\right]^{\frac{2}{\alpha+1}} (t-t_0)^{\frac{2}{\alpha+1}} \left(1 + O((t-t_0)^{\frac{2}{\alpha+1}})\right),$$

applying once more Proposition 5.2. to compute the last part of the formula above. This holds

$$u(t) = \left[\frac{\alpha+1}{2\alpha} \left(\frac{2\alpha^2}{\alpha-1}\right)^{\frac{1}{2}}\right]^{\frac{2}{\alpha+1}} (t-t_0)^{\frac{2}{\alpha+1}} + O((t-t_0)^{\frac{4}{\alpha+1}})$$

applying the product rule for "big-O" notation. We have found the Taylor expansion of u around t_0.

Now, we know that $\dot{u} = \frac{2}{\alpha+1} z^{\frac{1-\alpha}{\alpha+1}} \dot{z}$. We can follow the same steps as before to conclude that

$$\dot{z}(t) = \frac{\alpha+1}{2\alpha} \left(\frac{2\alpha^2}{\alpha-1}\right)^{\frac{1}{2}} + O((t-t_0)^{\frac{2}{\alpha+1}}).$$

Using the formula

$$z(t) = \frac{\alpha+1}{2\alpha} \left(\frac{2\alpha^2}{\alpha-1}\right)^{\frac{1}{2}} (t-t_0) \left[1 + O((t-t_0)^{\frac{2}{\alpha+1}})\right]$$

we deduce

$$\dot{u} = \frac{2}{\alpha+1} \left[\frac{\alpha+1}{\sqrt{2(\alpha-1)}} (t-t_0) \left(1 + O((t-t_0)^{\frac{2}{\alpha+1}})\right)\right]^{\frac{1-\alpha}{\alpha+1}} \left[\frac{\alpha+1}{\sqrt{2(\alpha-1)}} + O((t-t_0)^{\frac{2}{\alpha+1}})\right].$$

Applying Proposition 5.2 to expand $\left(1 + O((t-t_0)^{\frac{2}{\alpha+1}})\right)^{\frac{1-\alpha}{\alpha+1}}$, we obtain

$$\dot{u} = \frac{2}{\alpha+1} \left(\frac{\alpha+1}{\sqrt{2(\alpha-1)}}\right)^{\frac{1-\alpha}{\alpha+1}} (t-t_0)^{\frac{1-\alpha}{\alpha+1}} \left(1 + O((t-t_0)^{\frac{2}{\alpha+1}})\right) \left[\frac{\alpha+1}{\sqrt{2(\alpha-1)}} + O((t-t_0)^{\frac{2}{\alpha+1}})\right].$$

Let us denote

$$A = \frac{2}{\alpha+1}\left(\frac{\alpha+1}{\sqrt{2(\alpha-1)}}\right)^{\frac{1-\alpha}{\alpha+1}}$$
$$B = \frac{\alpha+1}{\sqrt{2(\alpha-1)}}.$$

Then

$$\dot{u} = A(t-t_0)^{\frac{1-\alpha}{\alpha+1}}\left[B + O((t-t_0)^{\frac{2}{\alpha+1}}) + BO((t-t_0)^{\frac{2}{\alpha+1}}) + O((t-t_0)^{\frac{2}{\alpha+1}})O((t-t_0)^{\frac{2}{\alpha+1}})\right].$$

It is easy to see that

$$O((t-t_0)^{\frac{2}{\alpha+1}})O((t-t_0)^{\frac{2}{\alpha+1}}) \subseteq O((t-t_0)^{\frac{4}{\alpha+1}}).$$

Also we can see in the proof of Proposition 5.2. that

$$O((t-t_0)^{\frac{4}{\alpha+1}}) \subseteq O((t-t_0)^{\frac{2}{\alpha+1}}).$$

Therefore,

$$\dot{u} = AB(t-t_0)^{\frac{1-\alpha}{\alpha+1}} + A(t-t_0)^{\frac{1-\alpha}{\alpha+1}}O((t-t_0)^{\frac{2}{\alpha+1}})).$$

Hence, we get

$$\dot{u} = \frac{2}{\alpha+1}\left(\frac{\alpha^2+2\alpha+1}{2(\alpha-1)}\right)^{\frac{1}{\alpha+1}}(t-t_0)^{\frac{1-\alpha}{\alpha+1}} + O\left((t-t_0)^{\frac{3-\alpha}{\alpha+1}}\right).$$

□

Proof. (of Propostion 1.3.) By the Fundamental Theorem of Calculus

$$h(t) = h(\tau) + \int_\tau^t p(s)\dot{u}(s)\ ds.$$

for all $t, \tau \in]t_0, t_1[$. Let $t \in]t_0, t_1[$ and define $h(t_0) := \lim_{\tau\to t_0} h(\tau)$ then

$$h(t_0) = h(t) - \int_{t_0}^t p(s)\dot{u}(s)\ ds,$$

where the integral is improper. Our goal is to prove that it converges for all $t \in]t_0, t_1[$. As p and $\dot{u}$ are continuous in $]t_0, t_1[$, we have just to check integrability near t_0. Let us write the expansion of $\dot{u}$ around t_0 as

$$\dot{u}(t) = C_0(t-t_0)^{\frac{1-\alpha}{\alpha+1}} + O((t-t_0)^{\frac{3-\alpha}{\alpha+1}}).$$

Then

$$\int_{t_0}^t p(s)\dot{u}(s)\ ds = \int_{t_0}^t C_0(s-t_0)^{\frac{1-\alpha}{\alpha+1}}p(s)\ ds + \int_{t_0}^t O((s-t_0)^{\frac{3-\alpha}{\alpha+1}})p(s)\ ds.$$

Notice that in the first integral as p is a bounded function there exists a number $M > 0$ such that

$$\int_{t_0}^{t} |C_0(s-t_0)^{\frac{1-\alpha}{\alpha+1}} p(s)|\ ds \leqslant \int_{t_0}^{t} M(s-t_0)^{\frac{1-\alpha}{\alpha+1}}\ ds.$$

Therefore we just have to check that

$$\int_{t_0}^{t} (s-t_0)^{\frac{1-\alpha}{\alpha+1}}\ ds$$

is integrable, which happens if the exponent $\frac{1-\alpha}{\alpha+1} > -1$. And in fact

$$\frac{1-\alpha}{\alpha+1} > \frac{-\alpha}{\alpha+1} = -\frac{1}{1+\frac{1}{\alpha}} > -1.$$

As p is bounded we know that

$$O((s-t_0)^{\frac{3-\alpha}{\alpha+1}})p(s) \subseteq O((s-t_0)^{\frac{3-\alpha}{\alpha+1}}).$$

By definition, for a function $f = O((s-t_0)^{\frac{3-\alpha}{\alpha+1}})$ there exists $\varepsilon > 0$ such that

$$|s-t_0| < \varepsilon\ \Rightarrow\ \frac{|f(s)|}{|s-t_0|^{\frac{3-\alpha}{\alpha+1}}} \leqslant K.$$

Consequently

$$\int_{t_0}^{t_0+\varepsilon} |f(s)|\ ds \leqslant \int_{t_0}^{t_0+\varepsilon} K|s-t_0|^{\frac{3-\alpha}{\alpha+1}}\ ds.$$

This integral converges as $\frac{3-\alpha}{\alpha+1} > -1$. $\square$

Let now $t_1 < +\infty$. Let us prove that t_1 is a collision point.

Proposition 1.8. *Let u be a maximal solution of (1) defined on $]t_0, t_1[$. Then t_1 is a collision point and*

$$\lim_{t\to t_1^-} \dot{u}(t) = -\infty.$$

Proof. Define the function

$$w(t) := u(-t) = u \circ R(t),$$

on $]-t_1, -t_0[$ where

$$R : \mathbb{R} \longrightarrow \mathbb{R},\ t \mapsto -t.$$

thus $R(]-t_1, -t_0[) =]t_0, t_1[$.

Differentiating w with respect to t we obtain

- $\dot{w}(t) = -\dot{u}(-t)$;
- $\ddot{w}(t) = \ddot{u}(-t)$.

Therefore w is a solution of the equation

$$\ddot{w} = -\frac{1}{w^{\alpha}(t)} + p \circ R(t),$$

where $p \circ R$ is bounded 2π-periodic, continuous, Lipschitz-continuous and differentiable if and only if p is bounded 2π-periodic, continuous, Lipschitz-continuous and differentiable, respectively.

Moreover w is defined in the maximal interval $]-t_1, -t_0[$, with $-t_1 > -\infty$. Then, by Proposition 1.2.,

$$\lim_{t \to -t_1^+} w(t) = 0.$$

Hence,

$$\lim_{t \to t_1^-} u(t) = \lim_{t \to t_1^-} w \circ R(t) = \lim_{t \to -t_1^+} w(t) = 0.$$

Also, by Lemma 1.5.,

$$\lim_{t \to -t_1^+} \dot{w}(t) = +\infty.$$

Thus,

$$\lim_{t \to t_1^-} \dot{u}(t) = \lim_{t \to t_1^-} -\dot{w} \circ R(t) = \lim_{t \to -t_1^+} -\dot{w}(t) = -\infty.$$

□

We say that the map R above is a *time reversing map*. Using this technique we can very easily prove analogous results to the previous for the point t_1, e.g., prove that the value of the energy function of the system at this point is finite.

1.2 Regularized system

Let u be a solution of (1) with maximal interval of definition $]t_0, t_1[$, where $t_0 > -\infty$. Let us define the integral

$$S(t) = \int_{t_0}^{t} \frac{1}{u^{\frac{\alpha}{2}}(s)}\, ds.$$

Proposition 1.9. *The function $S :]t_0, t_1[\longrightarrow \mathbb{R}$, called the* ***Sundman integral****, is well defined on its domain.*

Proof. Let us write the expansion of u around t_0 as

$$u(t) = C(t - t_0)^{\frac{2}{\alpha+1}} + O((t - t_0)^{\frac{4}{\alpha+1}}).$$

Then

$$u^{-\frac{\alpha}{2}}(t) = \tilde{C}(t - t_0)^{-\frac{\alpha}{\alpha+1}}[1 + O((t - t_0)^{\frac{2}{\alpha+1}})]^{-\frac{\alpha}{2}},$$

where we used the fact that if $f = O((t - t_0)^b)$ then $(t - t_0)^a f = O((t - t_0)^{a+b})$.

We can expand $[1 + O((t - t_0)^{\frac{2}{\alpha+1}})]^{-\frac{\alpha}{2}}$ as before using Proposition 5.2. to conclude that in a neighbourhood of t_0, say for $0 < t - t_0 < \varepsilon$,

$$u^{-\frac{\alpha}{2}}(t) = \tilde{C}(t - t_0)^{-\frac{\alpha}{\alpha+1}}[1 + O((t - t_0)^{\frac{2}{\alpha+1}})].$$

Therefore

$$\int_{t_0}^{t_0+\varepsilon} \frac{1}{u^{\frac{\alpha}{2}}(s)}\, ds = \tilde{C} \int_{t_0}^{t_0+\varepsilon} (s-t_0)^{-\frac{\alpha}{\alpha+1}} + (s-t_0)^{-\frac{\alpha}{\alpha+1}} O((s-t_0)^{\frac{2}{\alpha+1}})\, ds.$$

On one hand, the first integral

$$\int_{t_0}^{t_0+\varepsilon} (s-t_0)^{-\frac{\alpha}{\alpha+1}}\, ds$$

exists because the exponent $-\frac{\alpha}{\alpha+1} > -1$.

On the other hand, as

$$(s-t_0)^{-\frac{\alpha}{\alpha+1}} O((s-t_0)^{\frac{2}{\alpha+1}}) = O((s-t_0)^{\frac{2-\alpha}{\alpha+1}}),$$

using an argument we have already invoked in the last proof we can choose a smaller value for ε if necessary in order that

$$\int_{t_0}^{t_0+\varepsilon} O((s-t_0)^{\frac{2-\alpha}{\alpha+1}})\, ds$$

exists. Therefore S is well defined for all $t \in]t_0, t_1[$. $\square$

Observe that with help of the time reversing map we can also prove that the value $S(t_1)$ is well-defined at least if $t_1 < +\infty$. Hence S is well-defined in the interval $[t_0, t_1]$.

By the inverse function theorem, as the derivative of S is always non-zero in $]t_0, t_1[$, S has an inverse denoted by T and defined in an interval $]0, \sigma[$ with $\sigma > 0$

$$T :]0, \sigma[\longrightarrow]t_0, t_1[$$

which is continuous and of class C^1 on $]0, \sigma[$, again by the inverse function theorem.

Let us define $U(s) = u(T(s))$ and $H(s) = h(T(s))$ for every $s \in]0, \sigma[$. Computing the derivatives of U, H and T with respect to s we arrive to a new system

$$\begin{cases} U'' = \frac{1}{\alpha-1} + p(T)U^{\alpha} + \alpha U^{\alpha-1} H \\ T' = U^{\frac{\alpha}{2}} \\ H' = p(T)U' \end{cases} \tag{6}$$

which we call the *regularized system* for equation (1). Therefore we conclude that a solution of (1) generates a solution of (6), through the Sundman integral.

This autonomous system defines a continuous vector field on $\mathbb{R}^4$. Defining $V := U'$ we get the equation $x' = X(x)$ with

$$x = (U, V, T, H) \text{ and } X(x) = (V, \frac{1}{\alpha-1} + p(T)U^{\alpha} + \alpha U^{\alpha-1} H, U^{\frac{\alpha}{2}}, p(T)V).$$

Moreover as $\lim_{t\to t_0} u(t) = 0$ and $\lim_{t\to t_0} h(t) = h_0$ then u generates the solution of system (6) such that

$$U(0) = 0, \; V(0) = 0, \; T(0) = t_0, \text{ and } H(0) = h_0. \tag{7}$$

Conversely, if $x = (U, V, T, H)$ is a solution of system (6) then the function

$$I = U^\alpha H - \frac{1}{2}V^2 + \frac{1}{\alpha - 1}U$$

is constant over the solution of the system. In these conditions we say that I is an *integral of the motion*.

Moreover, if x satisfies the initial conditions in (7) then $U''(0) = \frac{1}{\alpha-1} > 0$. This implies that there exists a neighbourhood to the right of 0 in which $U(s) > 0$. Consequently, $T'(s) > 0$ in this neighbourhood and therefore T has a local inverse. Let us denote it by S and assume it is defined on an interval $]t_0, t_0 + \delta[$.

Now let us define $u(t) = U(S(t))$. Computing the derivative with respect to t we get

$$\ddot{u} = -\frac{1}{u^\alpha} + \frac{\alpha I}{u^{\alpha+1}} + p(t), \quad \forall t \in \;]t_0, t_0 + \delta[.$$

From the initial conditions (7) we deduce $I(0) = 0$ which implies $I(s) = 0 \; \forall s$. Then u satisfies equation (1) and

$$\lim_{t\to t_0^+} u(t) = 0$$

taking into account that if $T(0) = t_0$ then $S(t_0) = 0$. Also it is not difficult to prove using the function I that the energy function of the system satisfies $h(t) = H(S(t))$ and hence

$$\lim_{t\to t_0^+} h(t) = h_0.$$

To motivate the next result we make the following definition: we call *collision conditions* to the following

$$\begin{cases} \lim_{t\to t_0^+} u(t) = 0 \\ \lim_{t\to t_0^+} h(t) = h_0. \end{cases} \tag{8}$$

The first condition states the solution has a collision at t_0. The second gives the energy at the collision.

We will see in the next result that these conditions are enough to guarantee uniqueness of the classical solution of (1) and so they can replace the initial conditions we are used in the standard results.

Lemma 1.10. *(Existence and uniqueness of a maximal solution of (1) satisfying collision conditions)*

Let t_0, $h_0 \in \mathbb{R}$. There is a maximal solution of (1) defined in $]t_0, t_1[$ satisfying the collision conditions (8). *Moreover this solution is unique if p is Lipschitz continuous.*

Proof. Let t_0, $h_0 \in \mathbb{R}$. Consider the regularized system

$$\begin{cases} U' = V \\ V' = \frac{1}{\alpha-1} + p(T)U^\alpha + \alpha U^{\alpha-1}H \\ T' = U^{\frac{\alpha}{2}} \\ H' = p(T)V \end{cases}$$

Defining $x = (U, V, T, H)$ and denoting by X the vector field associated to this system as we have done before we get a system of the type $x' = X(s, x)$.

The vector field

$$X : \mathbb{R} \times \mathbb{R}^4 \longrightarrow \mathbb{R}^4$$

is continuous all over its domain and it satisfies the Lipschitz condition. Then, by the Cauchy-Peano theorem, given initial conditions $(s_0, x_0) = (0, (0, 0, t_0, h_0)) \in \mathbb{R}\times\mathbb{R}^4$ there exists a solution of the regularized system defined in an interval I containing $s_0 = 0$ in its interior. Moreover there is uniqueness for the solutions of the regular system. Considering the function $u(t) = U(S(t))$ as we did before we obtain a solution of (1) satisfying (8).

Suppose u_1 and u_2 are two solutions satisfying the collision conditions (8). Hence both these solutions generate solutions x_1 and x_2 of the regularized system with initial conditions $(0, (0, 0, t_0, h_0))$. Then $x_1 = x_2$. Proceeding backwards this holds $u_1 = u_2$ in a neighbourhood of t_0.

So far we have proved there exists a unique solution of (1) satisfying the collision conditions in (8) in an interval $]t_0, t_0 + \delta[$. Let u be this function.

If $\lim\limits_{t \to t_0+\delta^-} u(t) = 0$ then $]t_0, t_0 + \delta[$ is a maximal interval and the lemma is proved.

Otherwise, the solution may be extended to the right of $t_0 + \delta$. In this case the point $(t_0 + \delta, u(t_0 + \delta), \dot{u}(t_0 + \delta))$ is in the domain of the vector field associated to equation (1) and so there is a unique maximal solution with this initial condition. Then gluing together these two functions we get the solution we were looking for.

□

2 The generalized Cauchy problem

Definition 2.1. A *generalized* or *bouncing solution* of (1) is a continuous function $u : \mathbb{R} \longrightarrow [0, +\infty[$ satisfying

1. The set of instants of collision $Z = \{t \in \mathbb{R} : u(t) = 0\}$ is discrete.
2. For any open interval $I \subseteq \mathbb{R} \setminus Z$ the function u is in $C^2(I)$ and satisfies equation (1) on I.
3. For each $t_0 \in Z$ the limit
$$\lim_{t \to t_0} h(t)$$
exists.

In the third item of the definition above we take the limit of the energy function on both sides of t_0. This means that the energy function has a well defined value at t_0.

From now on we will distinguish two kinds of solutions for equation (1). Those with no collisions are the classical solutions opposite to those with one or more collisions which we have just defined to be bouncing solutions.

Next we wish to establish a result on the existence and uniqueness of bouncing solutions.

In order to do so we first need to prove some useful and also very important results.

2.1 Continuous Dependence

Let u be a classical solution of equation (1) with maximal interval of definition $]t_0, t_1[$, $t_0 > -\infty$ satisfying the collision conditions (8). Given $\varepsilon > 0$ such that $h_0 + \frac{1}{\alpha-1}\frac{1}{\varepsilon^{\alpha-1}} > 0$ consider the initial conditions

$$u(t_0) = \varepsilon, \quad \dot{u}(t_0) = +\sqrt{2\left(h_0 + \frac{1}{\alpha - 1}\frac{1}{\varepsilon^{\alpha-1}}\right)} \tag{9}$$

and let us denote by $u^\varepsilon(t)$ the solution of (1) satisfying these initial conditions.

Lemma 2.1. *Suppose that p is Lipschitz continuous and that $J \subseteq]t_0, t_1[$ is a compact interval. Then there is an $\varepsilon_J > 0$ such that if $0 < \varepsilon < \varepsilon_J$ then the solution u^ε is well defined and positive on J and*

$$\lim_{\varepsilon \to 0} u^\varepsilon(t) = u(t), \quad \lim_{\varepsilon \to 0} \dot{u}^\varepsilon(t) = \dot{u}(t)$$

uniformly on J.

Proof. Once again the technique of the proof is to reduce the problem to the regularized system. So let us again consider the system

$$\begin{cases} U' = V \\ V' = \frac{1}{\alpha-1} + p(T)U^\alpha + \alpha U^{\alpha-1}H \\ T' = U^{\frac{\alpha}{2}} \\ H' = p(T)V \end{cases}$$

which, we have seen, can be represented as a system $x' = X(s,x)$, with $X : \mathbb{R} \times \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ a continuous vector field satisfying the Lipschitz condition.

To the solution u^ε of (1) satisfying the initial conditions (9) is associated a solution of the regularized system, which we call $x^\varepsilon = (U^\varepsilon, V^\varepsilon, T^\varepsilon, H^\varepsilon)$, and satisfying the initial conditions

$$U^\varepsilon(0) = \varepsilon, \quad V^\varepsilon(0) = \varepsilon^{\frac{\alpha}{2}} \sqrt{2\left(h_0 + \frac{1}{\alpha - 1}\frac{1}{\varepsilon^{\alpha-1}}\right)}, \quad T^\varepsilon(0) = t_0, \quad H^\varepsilon(0) = h_0.$$

As before, we get this by defining

$$S^\varepsilon(t) = \int_{t_0}^{t} \frac{1}{(u^\varepsilon)^{\frac{\alpha}{2}}(s)}\, ds.$$

Next,we define T^ε as the inverse of S^ε in some neighbourhood of 0,

$$U^\varepsilon(s) = u^\varepsilon(T^\varepsilon(s))$$

and $H^\varepsilon(s) = h^\varepsilon(T^\varepsilon(s)))$, where h^ε is the energy function associated to the solution u^ε.

Let $J \subseteq]t_0, t_1[$ be a compact interval. Let us choose instants $\tau_1 < \tau_2 \ \in \ ]t_0, t_1[$ such that J is contained in $]t_0, \tau_1[$ and define $\sigma_1 = S(\tau_1)$ and $\sigma_2 = S(\tau_2)$.

We will prove now that the function x^ε is well defined in $[0, \sigma_2[$ for ε sufficiently small. Moreover, the solution $x^\varepsilon(s)$ converges uniformly to $x(s) = (U(s), V(s), T(s), H(s))$ for $s \in [0, \sigma_2[$.

We are dealing with the system $x' = X(s,x)$, for which there is uniqueness for the Cauchy problem. Recall that $x(s)$ is the solution to the Cauchy problem with initial conditions

$$x(0) = (0, 0, t_0, h_0)$$

and $x^\varepsilon(s)$ is the solution of the Cauchy problem with initial conditions

$$x^\varepsilon(0) = \left(\varepsilon, \varepsilon^{\frac{\alpha}{2}} \sqrt{2\left(h_0 + \frac{1}{\alpha - 1}\frac{1}{\varepsilon^{\alpha-1}}\right)}, t_0, h_0\right).$$

Choosing a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$, with $\varepsilon_n \searrow 0$, by Theorem 5.13. the solution x^{ε_n} converges uniformly to x in compact intervals where x is defined. Since $x(s)$ is well defined in an interval $[a,b] \subset [0, S(t_1)[$ containing $[0, \sigma_2[$ then for n sufficiently large so is x^{ε_n}, and it converges uniformly to x for $s \in [0, \sigma_2[$.

It is easy to see that this implies that, for sufficiently small ε, the solution x^ε is well defined in $[0, \sigma_2[$ and it converges uniformly to x on this interval.

Next we want to see that $U^\varepsilon(s) > 0, \quad \forall s \in [0, \sigma_2]$ for sufficiently small ε.

We know that for the solution x, $U''(0) = \frac{1}{\alpha-1}$. Therefore we can find $s_1 \in]0, \sigma_2]$ such that

$$U''(s) \geqslant \frac{1}{\alpha}, \quad \forall s \in [0, s_1].$$

As

$$(U^\varepsilon)'' = (V^\varepsilon)' = \frac{1}{\alpha - 1} + p(T^\varepsilon)(U^\varepsilon)^\alpha + \alpha(U^\varepsilon)^{\alpha-1}H^\varepsilon$$

and x^ε converges uniformly to x in $[0, \sigma_2]$, then $(U^\varepsilon)''$ converges uniformly to U'' on the same interval. Therefore we can find ε sufficiently small so that

$$(U^\varepsilon)''(s) \geqslant \frac{1}{2\alpha}, \quad \forall s \in [0, s_1].$$

By means of a Taylor expansion of U^ε around 0 we get what we wished since

$$U^\varepsilon(s) = \varepsilon + s\varepsilon^{\frac{\alpha}{2}}\sqrt{2\left(h_0 + \frac{1}{\alpha - 1}\frac{1}{\varepsilon^{\alpha-1}}\right)} + \frac{s^2}{2}(U^\varepsilon)''(\xi)$$

with $\xi \in [0, s_1]$, where this is some interval containing 0 and on which $(U^\varepsilon)''(\xi) \geqslant \frac{1}{2\alpha}$.

Hence, for $s \in [0, s_1]$ we get $U^\varepsilon(s) \geqslant \varepsilon$. On $[s_1, \sigma_2]$ we know that $U(s) > 0$. Thus as U^ε converges uniformly to U it follows that for sufficiently small ε the function U^ε will also be strictly positive on this interval.

And so we have proved that $U^\varepsilon(s) > 0$ for all $s \in [0, \sigma_2]$.

Now we are able to establish that $(T^\varepsilon)' = (U^\varepsilon)^{\frac{\alpha}{2}} > 0$ in the interval $[0, \sigma_2]$. And so the inverse function $S^\varepsilon = (T^\varepsilon)^{-1}$ is well defined and is differentiable in the interval $]t_0, \tau_1[$.

Now notice that S^ε converges uniformly to S in the interval $[t_0 + \delta, \tau_1]$ for $\delta > 0$ as small as we wish, by the second item in Proposition 5.3., since T^ε converges uniformly to T in $[0, \sigma_2]$.

Therefore the function $v^\varepsilon := U^\varepsilon \circ S^\varepsilon(t)$ converges uniformly to $u = U \circ S(t)$ on the interval $[t_0 + \delta, \tau_1]$. Let us check that v^ε is the solution u^ε of equation (1) that we seek.

By the same calculations made in the end of the previous section, when we differentiate v^ε with respect to t we get

$$\ddot{v}^\varepsilon = -\frac{1}{(v^\varepsilon)^\alpha(t)} + \frac{\alpha I}{(v^\varepsilon)^{\alpha+1}(t)} + p(t).$$

I is an integral of motion which means it is constant along solutions of the regularized system such as x^ε. As $I(0) = 0$, it is proved that v^ε is a solution of equation (1). Moreover, $v^\varepsilon(t_0) = \varepsilon$ and

$$\dot{v}^\varepsilon(t_0) = \dot{S}^\varepsilon(t_0)(U^\varepsilon)'(0) = \sqrt{2\left(h_0 + \frac{1}{\alpha - 1}\frac{1}{\varepsilon^{\alpha-1}}\right)}.$$

Therefore, as the solution of equation (1) with a given initial condition is unique, $v^\varepsilon = u^\varepsilon$ and we have proved that u^ε converges uniformly to u on J contained.

Also

$$\dot{u}^\varepsilon(t) = \dot{S}^\varepsilon(t)(U^\varepsilon)'(S^\varepsilon(t)) = \frac{(U^\varepsilon)'(S^\varepsilon(t))}{(U^\varepsilon)^{\frac{\alpha}{2}}(S^\varepsilon(t))}$$

and this function converges uniformly to

$$\frac{U'(S(t))}{U^{\frac{\alpha}{2}}(S(t))} = \dot{u}(t)$$

on J. $\square$

2.2 Comparison of solutions

On several results ahead we will have the need to compare solutions of similar equations or else to compare solutions of the same equation with different initial conditions. And we will need to tell if one of the solutions has a value greater than the other for all instants. These facts soon will play an important role.

In the following, we always consider the order in $\mathbb{R}^n$ defined in the Appendix.

Lemma 2.2. *Suppose $u_1(t)$ and $u_2(t)$ are classical solutions of (1) defined on maximal intervals $I_1 =]t_0, t_1[$ and $I_2 =]t_0^*, t_1^*[$ such that $I_1 \cap I_2 \neq \emptyset$. Suppose that for some $\tau \in I_1 \cap I_2$*

$$u_1(\tau) \leqslant u_2(\tau), \quad \dot{u}_1(\tau) \leqslant \dot{u}_2(\tau).$$

Then

$$t_1 \leqslant t_1^*, \quad u_1(t) \leqslant u_2(t), \quad \dot{u}_1 \leqslant \dot{u}_2(t), \quad \forall\, t \in [\tau, t_1[.$$

Proof. This can be seen applying Theorem 5.16. Consider the system

$$\begin{cases} \dot{u} = v \\ \dot{v} = -\frac{1}{u^\alpha} + p(t) \end{cases}$$

by reducing equation (1) to a first order system. It has the form $\dot{x} = X(t, x)$ with $x = (u, v)$. As X and its Jacobian matrix are continuous maps,

$$\frac{\partial X_1}{\partial v} = 1 \geqslant 0, \quad \frac{\partial X_2}{\partial u} = \frac{\alpha}{u^{\alpha+1}} \geqslant 0, \tag{10}$$

and $x_1 := (u_1, \dot{u}_1)$ and $x_2 := (u_2, \dot{u}_2)$ are two solutions of (1) satisfying

$$x_1(\tau) \leqslant x_2(\tau)$$

it follows that

$$x_1(t) \leqslant x_2(t), \quad \forall\, t \in [\tau, \min\{t_1, t_1^*\}].$$

It also follows that $\min\{t_1, t_1^*\} = t_1$, i.e., $t_1 \leqslant t_1^*$. $\square$

Lemma 2.3. *Suppose u_1 and u_2 are solutions of*

$$\ddot{u} = -\frac{1}{u^\alpha} + p_i(t)$$

for $i = 1, 2$, respectively, with p_1, p_2 Lipschitz-continuous and bounded functions satisfying

$$p_1(t) \leqslant p_2(t)$$

for each $t \in \mathbb{R}$ and u_1 and u_2 defined on maximal intervals $I_1 =]t_0, t_1[$ and $I_2 =]t_0^*, t_1^*[$ such that $I_1 \cap I_2 \neq \emptyset$. Suppose that for some $\tau \in I_1 \cap I_2$

$$u_1(\tau) \leqslant u_2(\tau), \quad \dot{u}_1(\tau) \leqslant \dot{u}_2(\tau).$$

Then

$$t_1 \leqslant t_1^*, \quad u_1(t) \leqslant u_2(t), \quad \dot{u}_1 \leqslant \dot{u}_2(t), \quad \forall\ t \in [\tau, t_1[.$$

Proof. Let

$$X(t, u, v) := (v, -\frac{1}{u^\alpha} + p_1(t)).$$

Let us define $v_1 := \dot{u}_1$ and $v_2 := \dot{u}_2$. By definition

$$(\dot{u}_1, \dot{v}_1) = X(t, u_1, v_1).$$

Since

$$(\dot{u}_2, \dot{v}_2) = \left(v_2, -\frac{1}{u_2^\alpha} + p_2(t)\right) \geqslant \left(v_2, -\frac{1}{u_2^\alpha} + p_1(t)\right),$$

it follows that

$$(\dot{u}_2, \dot{v}_2) \geqslant X(t, u_2, v_2).$$

Moreover,

$$u_1(\tau) \leqslant u_2(\tau), \text{ and } v_1(\tau) \leqslant v_2(\tau).$$

Applying Theorem 5.17, we may conclude that

$$t_1 \leqslant t_1^*, \quad u_1(t) \leqslant u_2(t), \quad \dot{u}_1 \leqslant \dot{u}_2(t), \quad \forall\ t \in [\tau, t_1[.$$

But first we need to be sure that X is of type-K, which is a simple exercise using (10) and the fundamental theorem of calculus. □

Lemma 2.4. *Suppose p_i are as in the previous lemma and u_i are solutions of*

$$\ddot{u} = -\frac{1}{u^\alpha} + p_i(t)$$

for $i = 1, 2$, defined in the maximal intervals $I_i =]t_0, t_i[$, respectively, with t_0 finite. Let

$$h_{0i} = \lim_{t \to t_0^+} \left[\frac{1}{2}\dot{u}_i^2(t) - \frac{1}{\alpha - 1}\frac{1}{u_i^{\alpha-1}(t)}\right], \quad i = 1, 2$$

be the energy function associated to the solutions u_i , respectively, at t_0.

If $h_{01} \leqslant h_{02}$ then

$$t_1 \leqslant t_2, \quad u_1(t) \leqslant u_2(t), \quad \dot{u}_1(t) \leqslant \dot{u}_2(t), \quad \forall\ t \in]t_0, t_1[.$$

Proof. Let

$$u_1^\varepsilon(t_0) = \varepsilon, \quad \dot{u}_1^\varepsilon = \sqrt{2\left(h_{01} + \frac{1}{\alpha - 1}\frac{1}{\varepsilon^{\alpha-1}}\right)}$$

and

$$u_2^\varepsilon(t_0) = \varepsilon, \quad \dot{u}_2^\varepsilon = \sqrt{2\left(h_{02} + \frac{1}{\alpha - 1}\frac{1}{\varepsilon^{\alpha-1}}\right)},$$

defined in $[t_0, t_1^\varepsilon[$ and $[t_0, t_2^\varepsilon[$, respectively.

Therefore,

$$u_1^\varepsilon(t_0) = u_2^\varepsilon(t_0), \quad \dot{u}_1^\varepsilon(t_0) \leqslant \dot{u}_2^\varepsilon(t_0).$$

Arguing as in the proof of Lemma 2.3., we conclude that

$$t_1^\varepsilon \leqslant t_2^\varepsilon, \quad u_1^\varepsilon(t) \leqslant u_2^\varepsilon(t), \quad \dot{u}_1^\varepsilon(t) \leqslant \dot{u}_2^\varepsilon(t),$$

for $t \in [t_0, t_1^\varepsilon[$. Since u_i^ε, with $i = 1, 2$, converges uniformly to u_i on compact intervals inside $]t_0, t_i[$, respectively, we conclude

$$t_1 \leqslant t_2, \quad u_1(t) \leqslant u_2(t), \quad \dot{u}_1(t) \leqslant \dot{u}_2(t),$$

for all $t \in]t_0, t_1[$. □

2.3 Existence and Uniqueness of Bouncing solution

Our goal in this section is to prove the following theorem

Theorem 2.5. *If p is Lipschitz-continuous, given $t_0, h_0 \in \mathbb{R}^2$ there exists a unique bouncing solution satisfying equation (1) and the collision conditions*

$$\begin{cases} \lim\limits_{t \to t_0^+} u(t) = 0 \\ \lim\limits_{t \to t_0^+} h(t) = h_0. \end{cases} \tag{11}$$

Before we proceed with the proof we must establish some facts about the particular case when the function p is constant.

2.3.1 Autonomous Equation

Consider the equation

$$\ddot{u}(t) = -\frac{1}{u^\alpha(t)} + P \tag{12}$$

where P is constant. This is an *autonomous equation*, i.e., (12) does not depend explicitly on time.

The main thing about this simpler equation is that the *mechanical energy of the solution* defined by

$$E(t) := E(u(t), \dot{u}(t)) = \frac{1}{2}\dot{u}^2(t) - \frac{1}{\alpha - 1}\frac{1}{u^{\alpha-1}(t)} - Pu(t)$$

is an integral of motion, i.e.,

$$\frac{d}{dt}E(t) \equiv 0$$

over the solutions of equation (12). Therefore, each solution has a constant mechanical energy, say E_0, such that

$$\frac{1}{2}\dot{u}^2(t) - \frac{1}{\alpha - 1}\frac{1}{u^{\alpha-1}(t)} - Pu(t) = E_0, \quad \forall\, t \in I$$

where I is the maximal interval of definition of u. We call *kinetic energy of the solution* to the function

$$K(\dot{u}(t)) = \frac{1}{2}\dot{u}^2(t)$$

and *potential energy of the solution* to

$$V(u(t)) = -\frac{1}{\alpha - 1}\frac{1}{u^{\alpha-1}(t)} - Pu(t),$$

so that

$$E(t) = K(t) + V(t).$$

The fact that $E(t)$ is constant allows us to deduce some relevant conclusions about the behaviour of this system. One of the most interesting consequences is that whenever $V(u(t))$ equals the mechanical energy E_0 it means $\dot{u}(t) = 0$. We are interested in these points because they may well be maxima for u.

1. If $P < 0$ then

   $$\lim_{u \to 0} V(u) = -\infty \quad \text{and} \quad \lim_{u \to +\infty} V(u) = +\infty.$$

   Differentiating V with respect to u we obtain

   $$\frac{d}{du}V(u) = \frac{1}{u^\alpha} - P > 0,$$

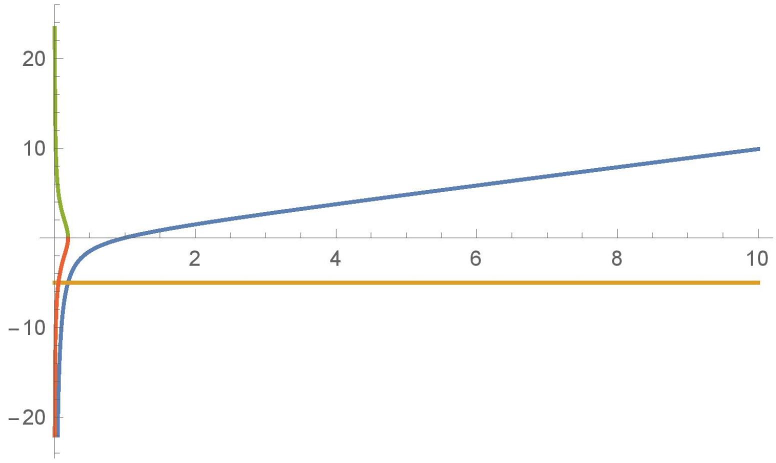
   thus $V(u)$ is strictly increasing and injective. Then we conclude that for every value E_0 there is one and only one value, denoted by u_{max}, for which

   $$V(u_{max}) = E_0$$

   (see Figure 1). Also the following limits hold

   $$E_0 \to +\infty \Rightarrow u_{max} \to +\infty, \qquad E_0 \to -\infty \Rightarrow u_{max} \to 0. \tag{13}$$

Figure 1: For $P = -1$: On the x-axis we have the values of u. The yellow line represents the mechanical energy E_0, the blue line represents the function $V(u)$, and the green and red ones are, respectively, the positive and negative values of $\dot{u}(u)$.



2. If $P > 0$ then

$$\lim_{u \to 0} V(u) = -\infty \quad \text{and} \quad \lim_{u \to +\infty} V(u) = -\infty.$$

Differentiating V with respect to u we obtain

$$\frac{d}{du} V(u) = \frac{1}{u^{\alpha}} - P.$$

Hence, defining $u^* = P^{-\frac{1}{\alpha}}$ it follows

$$\begin{cases} \frac{d}{du} V(u) > 0, & \text{if} \quad u < u^* \\ \frac{d}{du} V(u) = 0, & \text{if} \quad u = u^* \\ \frac{d}{du} V(u) < 0, & \text{if} \quad u > u^* \end{cases}$$

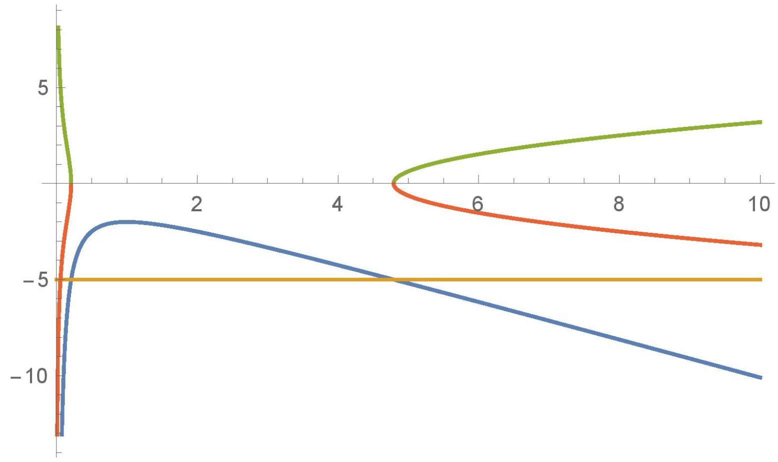
(See Figure 2). We set $V(u^*) := V_{max}$. In this case, the potential energy can attain the value E_0 if and only if $E_0 \leqslant V_{max}$, i.e.,

$$E_0 \leqslant -\frac{\alpha}{\alpha - 1} P^{\frac{\alpha-1}{\alpha}}.$$

As V is strictly increasing in $]0, u^*[$, there exists only one value of $u \in]0, u^*[$, which we call u_{max}, such that

$$V(u_{max}) = E_0.$$

Figure 2: For $P = 1$: On the x-axis we have the values of u. The yellow line represents the mechanical energy E_0, the blue line represents the function $V(u)$, and the green and red ones are, respectively, the positive and negative values of $\dot{u}(u)$.



3. If $P = 0$ then

$$\lim_{u \to 0} V(u) = -\infty \quad \text{and} \quad \lim_{u \to +\infty} V(u) = 0.$$

Also

$$\frac{d}{du} V(u) = \frac{1}{u^\alpha} > 0$$

(see Figure 3). We define u_{max} analogously whenever $E_0 < 0$.

We are interested on finding out what happens to the classical solution of equation (12), defined in $]t_0, t_1[$ and satisfying the collision conditions

$$\begin{cases} \lim\limits_{t \to t_0^+} u(t) = 0 \\ \lim\limits_{t \to t_0^+} h(t) = h_0. \end{cases}$$

Note that

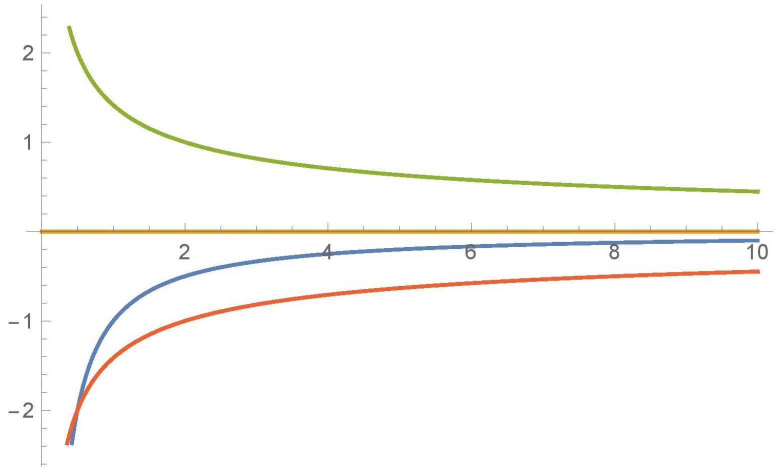
$$\lim_{t \to t_0^+} h(t) = \lim_{t \to t_0^+} E(t) = h_0$$

whenever $\lim\limits_{t \to t_0^+} u(t) = 0$ holds.

If the solution u assumes a value, u_{max}, such that

$$V(u_{max}) = h_0,$$

Figure 3: For $P = 0$: On the x-axis we have the values of u. The yellow line represents the mechanical energy E_0, the blue line represents the function $V(u)$, and the green and red ones are, respectively, the positive and negative values of $\dot{u}(u)$.



then there exists an instant, say $t_{max} \in]t_0, t_1[$, such that $u(t_{max}) = u_{max}$. Due to conservation of the mechanical energy of the solution, it follows

$$\dot{u}(t_{max}) = 0.$$

It is not hard to prove and we can find the proof on any mechanics book (cf. [11]) that the length of the maximal interval $t_1 - t_0$ of the solution of a conservative system is given by the formula

$$\tau(h_0, P) = 2\int_0^{u_{max}} \frac{du}{\sqrt{2\,(h_0 - V(u))}}.$$

Lemma 2.6. *Suppose $P < 0$. The following statements hold:*

1. *for each $h_0 \in \mathbb{R}$, $\tau(h_0, P) > 0$;*

2. $\lim_{h_0 \to -\infty} \tau(h_0, P) = 0$;

3. $\lim_{h_0 \to +\infty} \tau(h_0, P) = +\infty$.

Proof. 1. This item is trivial because we have an integral of a positive function.

2. Let h_0 be negative. Let us define

$$u_0 := \frac{u_{max}}{2}.$$

Therefore

$$-V(u_0) > -V(u_{max}) = -h_0,$$

i.e.,

$$1+(\alpha-1)Pu_0^\alpha+(\alpha-1)u_0^{\alpha-1}h_0>0$$

Let us split τ into two integrals

$$\tau(h_0,P)=2\int_0^{u_0}\frac{du}{\sqrt{2\left(h_0+Pu+\frac{1}{\alpha-1}\frac{1}{u^{\alpha-1}}\right)}}+2\int_{u_0}^{u_{max}}\frac{du}{\sqrt{2\left(h_0+Pu+\frac{1}{\alpha-1}\frac{1}{u^{\alpha-1}}\right)}}.$$

Let us focus first on the first integral, which we will refer to simply as A. For simplicity, we will omit the constant $\frac{2}{\sqrt{2}}$ for it will not take part in what follows.

$$A=\int_0^{u_0}\frac{du}{\sqrt{h_0+Pu+\frac{1}{\alpha-1}\frac{1}{u^{\alpha-1}}}}=\int_0^{u_0}\frac{\sqrt{(\alpha-1)u^{\alpha-1}}du}{\sqrt{(\alpha-1)u^{\alpha-1}h_0+(\alpha-1)Pu^\alpha+1}}$$

Notice that for each $u\in[0,u_0]$

$$\sqrt{(\alpha-1)u^{\alpha-1}}\leqslant\sqrt{(\alpha-1)u_0^{\alpha-1}}$$

and

$$(\alpha-1)u^{\alpha-1}h_0+(\alpha-1)Pu^\alpha+1\geqslant(\alpha-1)u_0^{\alpha-1}h_0+(\alpha-1)Pu_0^\alpha+1>0,$$

as h_0 and P are negative.

Therefore,

$$\begin{aligned}A &\leqslant\int_0^{u_0}\tfrac{\sqrt{(\alpha-1)u_0^{\alpha-1}}du}{\sqrt{(\alpha-1)u_0^{\alpha-1}h_0+(\alpha-1)Pu_0^\alpha+1}}\\ &=u_0\tfrac{\sqrt{(\alpha-1)u_0^{\alpha-1}}}{\sqrt{(\alpha-1)u_0^{\alpha-1}h_0+(\alpha-1)Pu_0^\alpha+1}}\end{aligned}$$

The fact that $-V(u_{max})=-h_0$ implies that

$$1+(\alpha-1)Pu_{max}^\alpha+(\alpha-1)h_0u_{max}^{\alpha-1}=0. \tag{14}$$

Then, using (14) we conclude that

$$(\alpha-1)h_0u_0^{\alpha-1}+(\alpha-1)Pu_0^\alpha+1=(\alpha-1)h_0\left(\frac{u_{max}}{2}\right)^{\alpha-1}+(\alpha-1)P\left(\frac{u_{max}}{2}\right)^\alpha+1=$$

$$\begin{aligned}&=\frac{1}{2^{\alpha-1}}\left(-1-(\alpha-1)Pu_{max}^\alpha\right)+(\alpha-1)P\left(\frac{u_{max}}{2}\right)^\alpha+1\\ &=1-\frac{1}{2^{\alpha-1}}-(\alpha-1)P\left(\frac{u_{max}}{2}\right)^\alpha\end{aligned}$$

So

$$A \leqslant \left(\frac{u_{max}}{2}\right)\frac{\sqrt{(\alpha-1)\left(\frac{u_{max}}{2}\right)^{\alpha-1}}}{\sqrt{1-\frac{1}{2^{\alpha-1}}-(\alpha-1)P\left(\frac{u_{max}}{2}\right)^{\alpha}}}$$

Notice that (13) implies that

$$h_0 \to +\infty \Rightarrow u_{max} \to +\infty, \quad h_0 \to -\infty \Rightarrow u_{max} \to 0$$

Therefore,

$$\lim_{h_0 \to -\infty} A = 0.$$

Now let us look to the second integral, to which we will simply refer to as B. Substituting h_0 by $V(u_{max})$ in B it follows

$$B = \int_{u_0}^{u_{max}} \frac{du}{\sqrt{P(u-u_{max})+\frac{1}{\alpha-1}\left(\frac{u_{max}^{\alpha-1}-u^{\alpha-1}}{u^{\alpha-1}u_{max}^{\alpha-1}}\right)}}.$$

Using Lagrange's Theorem on the function $u^{\alpha-1}$ we conclude

$$u^{\alpha-1} = u_{max}^{\alpha-1} + (\alpha-1)\xi_u^{\alpha-2}(u-u_{max}), \quad \text{with } \xi_u \in]u, u_{max}[. \tag{15}$$

Hence,

$$\frac{1}{\alpha-1}\left(\frac{u_{max}^{\alpha-1}-u^{\alpha-1}}{u^{\alpha-1}u_{max}^{\alpha-1}}\right) = \frac{u_{max}-u}{\xi_u^{2-\alpha}u_{max}^{2\alpha-2}+(\alpha-1)(u-u_{max})u_{max}^{\alpha-1}}.$$

As $P(u-u_{max}) \geqslant 0$ in $[u_0, u_{max}]$ one has

$$B \leqslant \int_{u_0}^{u_{max}} \frac{du}{\sqrt{\frac{1}{\alpha-1}\left(\frac{u_{max}^{\alpha-1}-u^{\alpha-1}}{u^{\alpha-1}u_{max}^{\alpha-1}}\right)}} = \int_{u_0}^{u_{max}} \sqrt{\frac{\xi_u^{2-\alpha}u_{max}^{2\alpha-2}+(\alpha-1)(u-u_{max})u_{max}^{\alpha-1}}{u_{max}-u}}du$$

As $u-u_{max} \leqslant 0$ in $[u_0, u_{max}]$ one has

$$B \leqslant \int_{u_0}^{u_{max}} \sqrt{\frac{\xi_u^{2-\alpha}u_{max}^{2\alpha-2}}{u_{max}-u}}du$$

Finally, as $\xi_u \geqslant u_0$ then $\xi_u^{2-\alpha} \leqslant u_0^{2-\alpha} = \left(\frac{u_{max}}{2}\right)^{2-\alpha}$. Thus,

$$\begin{aligned} B &\leqslant \int_{u_0}^{u_{max}} \sqrt{\frac{\left(\frac{u_{max}}{2}\right)^{2-\alpha} u_{max}^{2\alpha-2}}{u_{max}-u}}du = 2^{\frac{\alpha-2}{2}}\int_{u_0}^{u_{max}} \sqrt{\frac{u_{max}^{\alpha}}{u_{max}-u}}du \\ &= 2^{\frac{\alpha-2}{2}}\sqrt{u_{max}^{\alpha}}\left[2(u_{max}-u)^{1/2}\right]_{u_0}^{u_{max}} = 2^{\frac{\alpha-2}{2}}\sqrt{u_{max}^{\alpha}}2\left(\frac{u_{max}}{2}\right)^{1/2} \end{aligned}$$

Therefore, it is obvious that $\lim_{h_0 \to -\infty} B = 0$.

Hence, $\lim_{h_0 \to -\infty} \tau(h_0, P) = 0$.

3. Let h_0 be positive and consider some $c \in]0, u_{max}[$.

$$\int_0^c \frac{du}{\sqrt{h_0 + Pu + \frac{1}{\alpha-1}\frac{1}{u^{\alpha-1}}}} \geqslant \int_0^c \frac{du}{\sqrt{h_0 + \frac{1}{\alpha-1}\frac{1}{u^{\alpha-1}}}} = \int_0^c \frac{\sqrt{(\alpha-1)u^{\alpha-1}}}{\sqrt{(\alpha-1)h_0 u^{\alpha-1}+1}} du$$

The first inequality is legitimate because

$$h_0 + \frac{1}{\alpha-1}\frac{1}{u^{\alpha-1}} > h_0 + Pu + \frac{1}{\alpha-1}\frac{1}{u^{\alpha-1}} > 0.$$

As u is being integrated between 0 and c we can proceed with the inequalities:

$$\begin{aligned} \geqslant \int_0^c \frac{\sqrt{(\alpha-1)u^{\alpha-1}}du}{\sqrt{(\alpha-1)h_0 c^{\alpha-1}+1}} &= \frac{1}{\sqrt{(\alpha-1)h_0 c^{\alpha-1}+1}} \left[u^{\frac{\alpha+1}{2}}\right]_0^c \\ &= \frac{c^{\frac{\alpha+1}{2}}}{\sqrt{(\alpha-1)h_0 c^{\alpha-1}+1}}. \end{aligned}$$

Therefore, when $h_0 \to +\infty$ this integral approaches zero.

Writing down $h_0 = -Pu_{max} - \frac{1}{\alpha-1}\frac{1}{u_{max}^{\alpha-1}}$, then

$$\int_c^{u_{max}} \frac{du}{\sqrt{P(u-u_{max}) + \frac{1}{\alpha-1}\left(\frac{1}{u^{\alpha-1}} - \frac{1}{u_{max}^{\alpha-1}}\right)}}$$

$$= \int_c^{u_{max}} \frac{du}{\sqrt{P(u-u_{max}) + \frac{1}{\alpha-1}\left(\frac{u_{max}^{\alpha-1} - u^{\alpha-1}}{u^{\alpha-1}u_{max}^{\alpha-1}}\right)}}$$

Using (15) the above integral becomes

$$\int_c^{u_{max}} \frac{du}{\sqrt{P(u-u_{max}) - \frac{\xi_u^{\alpha-2}(u-u_{max})}{u^{\alpha-1}u_{max}^{\alpha-1}}}} =$$

$$= \int_c^{u_{max}} \frac{du}{\sqrt{\left(\frac{\xi_u^{\alpha-2}}{u^{\alpha-1}u_{max}^{\alpha-1}} - P\right)(u_{max}-u)}}$$

As u is being integrated between c and u_{max} it follows

$$\frac{\xi_u^{\alpha-2}}{u^{\alpha-1}u_{max}^{\alpha-1}} \leqslant \frac{u_{max}^{\alpha-2}}{c^{\alpha-1}u_{max}^{\alpha-1}} = \frac{1}{c^{\alpha-1}u_{max}}$$

and we can proceed with the inequalities for the integral

$$\geqslant \int_c^{u_{max}} \frac{du}{\sqrt{\left(\frac{1}{c^{\alpha-1}u_{max}} - P\right)(u_{max}-u)}} = \frac{2\sqrt{u_{max}-c}}{\sqrt{\frac{1}{c^{\alpha-1}u_{max}} - P}}$$

Letting $h_0 \to +\infty$ we know that $u_{max} \to +\infty$. Therefore

$$\lim_{h_0\to+\infty} \frac{2\sqrt{u_{max}-c}}{\sqrt{\frac{1}{c^{\alpha-1}u_{max}} - P}} = +\infty.$$

This implies that $\lim\limits_{h_0\to+\infty} \tau(h_0,P) = +\infty$.

□

For later use in the final section we will need also to discuss the behaviour of the function τ when $P > 0$ and as h_0 approaches $-\infty$.

Lemma 2.7. *Suppose $P > 0$. Then $\lim\limits_{h_0\to-\infty} \tau(h_0,P) = 0$.*

Proof. Let us choose

$$h_0 \leqslant V_{max} := -\frac{\alpha}{\alpha-1}P^{\frac{\alpha-1}{\alpha}} < 0.$$

As the potential energy V is strictly increasing on the interval $]0, P^{-\frac{1}{\alpha}}[$, there is one and only one $u \in]0, P^{-\frac{1}{\alpha}}[$, denoted by u_{max}, such that

$$V(u_{max}) = h_0.$$

We will prove that

$$\int_0^{u_{max}} \frac{du}{\sqrt{h_0 + Pu + \frac{1}{\alpha-1}\frac{1}{u^{\alpha-1}}}} \longrightarrow 0$$

as $h_0 \to -\infty$.

Define the function

$$\widetilde{V}(u) := -\frac{1}{\alpha-1}\frac{1}{u^{\alpha-1}}.$$

As P is positive,

$$V(u) < \widetilde{V}(u), \quad \forall\ u \in]0,+\infty[.$$

Moreover this function is strictly increasing on $]0,+\infty[$ as its derivative is positive

$$\frac{d}{du}\widetilde{V} = \frac{1}{u^{\alpha}} > 0.$$

Therefore there exists $u_0 \in]0,+\infty[$ such that

$$\widetilde{V}(u_0) = h_0.$$

However as $h_0 = V(u_{max})$ and $V(u_0) < \widetilde{V}(u_0)$, it follows that

$$V(u_0) < V(u_{max})$$

and therefore

$$u_0 < u_{max}. \tag{16}$$

On one hand we know that

$$\frac{1}{\alpha - 1}\frac{1}{u_0^{\alpha-1}} = -h_0.$$

Thus

$$u_0^{\alpha-1} = -\frac{1}{\alpha - 1}\frac{1}{h_0}. \tag{17}$$

On the other hand as

$$-Pu_{max} - \frac{1}{\alpha - 1}\frac{1}{u_{max}^{\alpha-1}} = h_0$$

it follows that

$$u_{max}^{\alpha-1} = -\frac{1}{(\alpha - 1)[h_0 + Pu_{max}]}.$$

Consequently if we divide these values, we get

$$\frac{u_0^{\alpha-1}}{u_{max}^{\alpha-1}} = 1 + \frac{Pu_{max}}{h_0}.$$

By letting $h_0 \to -\infty$ and noting that

$$\lim_{h_0 \to -\infty} u_{max} = 0 \tag{18}$$

we conclude that

$$\frac{u_0^{\alpha-1}}{u_{max}^{\alpha-1}} \longrightarrow 1. \tag{19}$$

$$\int_0^{u_0/2} \frac{du}{\sqrt{h_0 + Pu + \frac{1}{\alpha-1}\frac{1}{u^{\alpha-1}}}} \leqslant \int_0^{u_0/2} \frac{du}{\sqrt{h_0 + \frac{1}{\alpha-1}\frac{1}{u^{\alpha-1}}}} = \star,$$

taking into account that when $u \in]0, u_0[$

$$\frac{1}{\alpha - 1}\frac{1}{u^{\alpha-1}} > \frac{1}{\alpha - 1}\frac{1}{u_0^{\alpha-1}} = -h_0$$

and thus

$$h_0 + \frac{1}{\alpha - 1}\frac{1}{u^{\alpha-1}} > 0.$$

We can proceed with the inequalities above

$$\star \leqslant \int_0^{u_0/2} \frac{du}{\sqrt{h_0 + \frac{1}{\alpha-1}\frac{2^{\alpha-1}}{u_0^{\alpha-1}}}} = \int_0^{u_0/2} \frac{\sqrt{(\alpha-1)(u_0/2)^{\alpha-1}}du}{\sqrt{(\alpha-1)h_0(u_0/2)^{\alpha-1}+1}} =$$

$$= \frac{\sqrt{(\alpha-1)(u_0/2)^{\alpha-1}}}{\sqrt{(\alpha-1)h_0(u_0/2)^{\alpha-1}+1}}\frac{u_0}{2} = \sqrt{\frac{\alpha-1}{1-\frac{1}{2^{\alpha-1}}}}\left(\frac{u_0}{2}\right)^{\frac{\alpha+1}{2}},$$

where the last equality uses (17).

Once we let $h_0 \to -\infty$ and noting (18) and (16) we get

$$\sqrt{\frac{\alpha-1}{1-\frac{1}{2^{\alpha-1}}}}\left(\frac{u_0}{2}\right)^{\frac{\alpha+1}{2}} \longrightarrow 0$$

As for the other integral

$$\int_{u_0/2}^{u_{max}} \frac{du}{\sqrt{h_0 + Pu + \frac{1}{\alpha-1}\frac{1}{u^{\alpha-1}}}} = \int_{u_0/2}^{u_{max}} \frac{du}{\sqrt{P(u-u_{max}) + \frac{1}{\alpha-1}\left(\frac{u_{max}^{\alpha-1}-u^{\alpha-1}}{u_{max}^{\alpha-1}u^{\alpha-1}}\right)}}.$$

Again using (15) the integral becomes

$$\int_{u_0/2}^{u_{max}} \frac{du}{\sqrt{\left(\frac{\xi_u^{\alpha-2}}{u_{max}^{\alpha-1}u^{\alpha-1}} - P\right)(u_{max}-u)}}.$$

We need to be sure that

$$\frac{\xi_u^{\alpha-2}}{u_{max}^{\alpha-1}u^{\alpha-1}} - P \geqslant 0$$

in $]\frac{u_0}{2}, u_{max}[$. In fact, as $\xi_u \geqslant \frac{u_0}{2}$ and $u \leqslant u_{max}$ on this interval then

$$\frac{\xi_u^{\alpha-2}}{u_{max}^{\alpha-1}u^{\alpha-1}} - P \geqslant \frac{1}{2}\frac{u_0^{\alpha-2}}{u_{max}^{\alpha-2}}\frac{1}{u_{max}^{\alpha}} - P = \frac{1}{2}\frac{u_{max}}{u_0}\frac{u_0^{\alpha-1}}{u_{max}^{\alpha-1}}\frac{1}{u_{max}^{\alpha}} - P$$

As $u_{max} > u_0$, the above becomes

$$> \frac{1}{2}\frac{u_0^{\alpha-1}}{u_{max}^{\alpha-1}}\frac{1}{u_{max}^{\alpha}} - P \longrightarrow +\infty,$$

where the limit is taking by letting $h_0 \to -\infty$ and using (19). Thus by choosing h_0 sufficiently negative, there is a $K > 0$ such that

$$\frac{\xi_u^{\alpha-2}}{u_{max}^{\alpha-1}u^{\alpha-1}} - P \geqslant K > 0.$$

Consequently the integral above is

$$\leqslant \int_{u_0/2}^{u_{max}} \frac{du}{\sqrt{K(u_{max}-u)}} = \frac{2}{\sqrt{K}}\left(u_{max} - \frac{u_0}{2}\right)^{\frac{1}{2}} \longrightarrow 0,$$

where the limit is taken when $h_0 \to -\infty$. $\square$

2.3.2 Proof of Theorem 2.5.

Proof. We know by Lemma 1.10., there is a unique classical maximal solution u satisfying the collision conditions (8), defined in the interval $]t_0, t_1[$.

In order to obtain a bouncing solution we will glue together classical solutions at collision instants to obtain a continuous function u which will be the bouncing solution. At the first instant of collision t_1 we define

$$u(t_1) = 0$$

and

$$\lim_{t \to t_1^+} h(t) = \lim_{t \to t_1^-} h(t).$$

Then we glue together the classical solutions with collision conditions (t_0, h_0) and (t_1, h_1) and repeat this process recursively over all instants of collision. To glue together solutions defined in maximal intervals below t_0 we reduce this to the first case using the time reversing map.

Now we must verify that by proceeding like this we will obtain a function defined in the whole $\mathbb{R}$, because it could happen that the instants of collision accumulate at finite time.

Let us reason by contradiction. Suppose u is the function obtained from gluing together classical solutions as described previously. Let us say that the collisions occur in an increasing sequence of instants $\{t_n\}_{n \in \mathbb{N}_0}$ such that

$$u(t_n) = 0, \quad \lim_{t \to t_n} h(t) := h_n. \tag{20}$$

Suppose this sequence is bounded. Thus, the function u accumulates at a finite time and, hence, is not defined for all $\mathbb{R}$.

Comparing the solutions of the equations

$$\ddot{u} = -\frac{1}{u^\alpha} + p(t)$$

and

$$\ddot{u} = -\frac{1}{u^\alpha} - \|p\|_\infty$$

satisfying the collision conditions in (20), we conclude that

$$\tau(h_n, -\|p\|_\infty) \leqslant t_{n+1} - t_n$$

Hence,

$$\sum_{n \in \mathbb{N}_0} \tau(h_n, -\|p\|_\infty) \leqslant \sum_{n \in \mathbb{N}_0} (t_{n+1} - t_n).$$

Since

$$\sum_{n \in \mathbb{N}_0} (t_{n+1} - t_n) = \lim_{n \to +\infty} t_n - t_0 < +\infty,$$

it follows that $\sum_{n \in \mathbb{N}_0} \tau(h_n, -\|p\|_\infty)$ is a convergent series. Thus,

$$\lim_{n \to +\infty} \tau(h_n, -\|p\|_\infty) = 0$$

And so, Lemma 2.6 implies that $\lim\limits_{n\to+\infty} h_n = -\infty$.

As $\dot{h}(t) = p(t)\dot{u}(t)$, from the fundamental theorem of calculus we derive

$$h_{n+1} - h_n = \int_{t_n}^{t_{n+1}} p(s)\dot{u}(s)\ ds$$

Notice that as p is Lipschitz-continuous it is absolutely continuous. Therefore its derivative is defined almost everywhere and it is bounded, i.e.,

$$\dot{p}(t) \leqslant \|\dot{p}\|_\infty := \sup_{t\in I} \dot{p}(t),$$

where I is the set where $\dot{p}$ is defined. Therefore

$$h_{n+1} - h_n = -\int_{t_n}^{t_{n+1}} \dot{p}(s)u(s)\ ds.$$

Define the map, which we will call u_{max}, such that to each value h_0 in the collision conditions and to each value P of the autonomous equation assigns the number $u_{max}(h_0, P)$ which is the corresponding value u_{max} defined previously.

Comparing the solutions of the equations

$$\ddot{u} = -\frac{1}{u^\alpha} + p(t)$$

and

$$\ddot{u} = -\frac{1}{u^\alpha} + \|p\|_\infty$$

we conclude

$$\max_{[t_n,t_{n+1}]} u(t) \leqslant u_{max}(h_n, \|p\|_\infty),$$

and remember that $\lim\limits_{h_n\to-\infty} u_{max}(h_n, \|p\|_\infty) = 0$.

Thus,

$$|h_{n+1} - h_n| \leqslant \|\dot{p}\|_\infty u_{max}(h_n, \|p\|_\infty)|t_{n+1} - t_n|.$$

As for sufficiently large values of n, $u_{max}(h_n, \|p\|_\infty) < 1$ it follows that

$$|h_{n+1} - h_n| \leqslant \|\dot{p}\|_\infty |t_{n+1} - t_n|,$$

for sufficiently large values of n. Then

$$\sum_{n\in\mathbb{N}_0} |h_{n+1} - h_n| \leqslant \|\dot{p}\|_\infty \sum_{n\in\mathbb{N}_0} |t_{n+1} - t_n|.$$

Hence, $\sum_{n\in\mathbb{N}_0} |h_{n+1} - h_n|$ is convergent and so it follows that $\sum_{n\in\mathbb{N}_0} h_{n+1} - h_n$ also is.

However,

$$\sum_{n\in\mathbb{N}_0} h_{n+1} - h_n = \lim_{n\to+\infty} h_n - h_0 = -\infty.$$

This is a contradiction. Therefore the infinite sequence $\{t_n\}_{n\in\mathbb{N}}$ cannot accumulate at a finite time. $\square$

3 The dynamics in the plane

3.1 The successor map

From now on p is 2π-periodic and Lipschitz-continuous.

By Theorem 2.5, given $(t_0, h_0) \in \mathbb{R}^2$ there is only one bouncing solution satisfying the collision conditions in (8). We denote this solution by $u(t; t_0, h_0)$. Let $t_1 > t_0$ be the next instant of collision. If t_1 is finite, let us denote by h_1 the corresponding value of the energy at this point.

We define the *successor map* as

$$\mathcal{P} : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \qquad \mathcal{P}(t_0, h_0) = (t_1, h_1)$$

where $D = \{(t_0, h_0) \in \mathbb{R}^2 : \; t_1 < +\infty\}$.

Proposition 3.1. *Let $(t_0, h_0) \in \mathbb{R}^2$. We have that*

$$u(t; t_0 + 2\pi, h_0) = u(t - 2\pi; t_0, h_0), \;\; \forall \; t \in \mathbb{R}.$$

Proof. The function $u(t; t_0 + 2\pi, h_0)$ is the solution of equation (1), such that

$$\lim_{t \to t_0 + 2\pi} h(t) = h_0.$$

Consider the function

$$v(t) = u(t - 2\pi; t_0, h_0) = u(\dot{;} t_0, h_0) \circ \lambda_{-2\pi}(t)$$

where $\lambda_{-2\pi}$ is a translation by -2π

$$\lambda_{-2\pi} : \mathbb{R} \longrightarrow \mathbb{R}, \;\; t \mapsto t - 2\pi.$$

Computing the first and second derivatives of v with respect to time

$$\dot{v}(t) = \dot{u}(t - 2\pi; t_0, h_0)$$

$$\ddot{v} = \ddot{u}(t - 2\pi; t_0, h_0)$$

Therefore v satisfies

$$\ddot{v}(t) = -\frac{1}{v^\alpha(t)} + p(t)$$

as p is 2π-periodic. Therefore v is a solution of equation (1). Moreover

$$\lim_{t \to t_0 + 2\pi} h(v(t), \dot{v}(t)) = \lim_{t \to t_0 + 2\pi} \left(\frac{1}{2}\dot{v}^2(t) - \frac{1}{\alpha - 1} \frac{1}{v^{\alpha - 1}(t)} \right)$$

$$= \lim_{t \to t_0 + 2\pi} \left(\frac{1}{2}\dot{u}^2(t - 2\pi; t_0, h_0) - \frac{1}{\alpha - 1} \frac{1}{u^{\alpha - 1}(t - 2\pi; t_0, h_0)} \right)$$

$$\lim_{s\to t_0}\left(\frac{1}{2}\dot{u}^2(s;t_0,h_0)-\frac{1}{\alpha-1}\frac{1}{u^{\alpha-1}(s;t_0,h_0)}\right)$$
$$=\lim_{s\to t_0} h(u(s),\dot{u}(s))=h_0$$

Therefore v is the solution of (1) with collision conditions $(t_0+2\pi,h_0)$. Hence

$$v(t)=u(t;t_0+2\pi,h_0).$$

□

Lemma 3.2. *The integral*

$$S(t;t_0,h_0)=\int_{t_0}^{t}\frac{ds}{u^{\frac{\alpha}{2}}(s;t_0,h_0)}$$

is a continuous function of the three variables in the set

$$\mathcal{D}=\{(t;t_0,h_0)\in\mathbb{R}^3:\ t_0<t<t_1\}.$$

Proof. We need again to consider the regularized system

$$\begin{cases} U'=V\\ V'=\frac{1}{\alpha-1}+p(T)U^{\alpha}+\alpha U^{\alpha-1}H\\ T'=U^{\frac{\alpha}{2}}\\ H'=p(T)V\end{cases}$$

satisfying the initial conditions

$$U(0)=0,\ V(0)=0,\ T(0)=t_0,\ \text{and } H(0)=h_0.$$

Let us denote the solution to this problem by

$$x(s;t_0,h_0)=(U(s;t_0,h_0),V(s;t_0,h_0),T(s;t_0,h_0),H(s;t_0,h_0)).$$

i) Let us prove that $x(s;t_0,h_0)$ is continuous in the three variables.

This is consequence of theorem 5.14 in the Appendix. The solution $x(s;s_0,x_0)$ is continuous in the set

$$\{(s;s_0,x_0)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}^4:\ s\in I_{(s_0,x_0)}\}.$$

As $x(s;t_0;h_0)=x(s;0,(0,0,t_0,h_0))$, we conclude that it is continuous in these three variables.

ii) Let $\{(t_{0n}, h_{0n})\}_{n\in\mathbb{N}}$ be a sequence converging to (t_0, h_0).

We are going to prove that for all n sufficiently large $U(s; t_{0n}, h_{0n}) > 0$ in $]0, \sigma]$.

First notice that we can choose $\sigma > 0$ such that $U(s; t_0, h_0) > 0$ in $]0, \sigma]$. Also, we proved in Lemma 2.1 that

$$U(s; t_{0n}, h_{0n}) \longrightarrow U(s; t_0, h_0) \;\text{ uniformly on }\; [0, \sigma].$$

As $U''(0) = \frac{1}{\alpha-1}$, there exists $s_1 \in]0, \sigma]$ such that $U''(s) \geqslant \frac{1}{\alpha}$ for all $s \in [0, s_1]$.

Then uniform convergence implies that we can find n sufficiently large such that $U''(s; t_{0n}, h_{0n}) \geqslant \frac{1}{2\alpha}$ on $[0, s_1]$.

Using a Taylor expansion we find that in a neighbourhood of zero $[0, \varepsilon[$

$$U(s; t_{0n}, h_{0n}) = U(0; t_{0n}, h_{0n}) + sU'(0; t_{0n}, h_{0n}) + \frac{s^2}{2} U''(\xi_s; t_{0n}, h_{0n})$$

where $\xi_s \in]0, s[$. Set $\bar{s} = \min\{s_1, \varepsilon\}$. Then

$$U(s; t_{0n}, h_{0n}) > 0, \quad \forall s \in]0, \bar{s}[.$$

In what concerns the interval $[\bar{s}, \sigma]$ as $U(s; t_0, h_0) > 0$ on this interval we can find n sufficiently large such that $U(s; t_{0n}, h_{0n}) > 0$ on $[\bar{s}, \sigma]$ for all indices greater than n.

Hence,

$$U(s; t_{0n}, h_{0n}) > 0, \quad \forall s \in]0, \sigma]. \tag{21}$$

iii) Finally we are ready to prove the continuity of S on $\mathcal{D}$.

Let $\{(t_n; t_{0n}, h_{0n})\}_{n\in\mathbb{N}} \subseteq \mathcal{D}$ be a convergent sequence and let it converge to $(t^*; t_0, h_0) \in \mathcal{D}$.

We want to prove that

$$s_n := S(t_n; t_{0n}, h_{0n}) \to s^* := S(t^*; t_0, h_0)$$

which implies the continuity of S. We will accomplish this in three steps.

(a) First we prove that the sequence $\{s_n\}_{n\in\mathbb{N}}$ is bounded.

Observe that

$$\begin{cases} T(s_n; t_{0n}, h_{0n}) = t_n \\ T(s^*; t_0, h_0) = t^*. \end{cases}$$

From the regularized system we know that when U is positive, T is strictly increasing because

$$T'(s; t_0, h_0) = U^{\frac{\alpha}{2}}(s; t_0, h_0).$$

Assume that $s^* \in]0,\sigma[$. Otherwise we could choose σ large enough so this would happen. As T is strictly increasing in the interval $]0,\sigma]$, we deduce that there exists $\delta > 0$ such that

$$T(\sigma; t_0, h_0) = t^* + \delta,$$

with $t^* + \delta < t_1$.

As

$$T(s; t_{0n}, h_{0n}) \longrightarrow T(s; t_0, h_0) \text{ uniformly on } [0,\sigma],$$

we can find n sufficiently large such that

$$T(\sigma; t_{0n}, h_{0n}) \in \left] T(\sigma; t_0, h_0) - \frac{\delta}{2}, T(\sigma; t_0, h_0) + \frac{\delta}{2} \right[.$$

Hence

$$T(\sigma; t_{0n}, h_{0n}) > t^* + \frac{\delta}{2},$$

for all sufficiently large n.

Moreover, as $t_n \to t^*$ we can also find n sufficiently large so that

$$t_n < t^* + \frac{\delta}{2}.$$

Consequently, for all n sufficiently large it follows

$$T(\sigma; t_{0n}, h_{0n}) > t^* + \frac{\delta}{2} > t_n = T(s_n; t_{0n}, h_{0n}).$$

Now, it follows from (21) that $T(\cdot; t_{0n}, h_{0n})$ is also strictly increasing in $[0,\sigma]$, this holds $s_n < \sigma$ for all n sufficiently large.

Therefore, $\{s_n\}$ is a bounded sequence with $s_n \in [0,\sigma]$ for all sufficiently large n.

(b) In the second step we prove that every convergent subsequence of s_n converges to s^*. Denote by $\{s_k\}$ be a convergent subsequence of $\{s_n\}$. Note that such a sequence exists by compactness. Let

$$s_k \to s' \in [0,\sigma].$$

Hence, by continuity

$$\begin{cases} T(s_k; t_0, h_0) \to T(s'; t_0, h_0) \\ T(s_k; t_0, h_0) = t_k \to t^* \end{cases}$$

Therefore,

$$T(s'; t_0, h_0) = t^*.$$

As T is strictly increasing in $[0,\sigma]$, T is thereby injective. Thus, $s' = s^*$.

(c) Final step we conclude that the whole sequence converges to s^*.
By contradiction, if this was not the case, then by definition it would happen that

$$\exists\, \delta > 0 \ \ \forall N > 0 \ \ \exists n_N > N : \ \ |s_{n_N} - s^*| > \delta.$$

Then we can construct a subsequence denoted by $\{s_{n_N}\}$ that does not have subsequences converging to s^*. But as $\{s_{n_N}\}$ is bounded then, by compactness, it has convergent subsequences, which are also convergent subsequences of $\{s_n\}$. This is a contradiction.
Thus, $s_n \to s^*$. □

Lemma 3.3. *The map*

$$(t; t_0, h_0) \in \mathcal{D} \mapsto (u(t; t_0, h_0), \dot{u}(t; t_0, h_0)) \in \mathbb{R}^2$$

is continuous.

Proof. Let us observe that

$$u(t; t_0, h_0) = U(S(t; t_0, h_0); t_0, h_0) = U \circ S(t; t_0, h_0)$$

and

$$\dot{u}(t; t_0, h_0) = \dot{S}(t; t_0, h_0) U'(S(t; t_0, h_0); t_0, h_0) = \frac{U'(S(t; t_0, h_0); t_0, h_0)}{U^{\frac{\alpha}{2}}(S(t; t_0, h_0); t_0, h_0)},$$

hence

$$\dot{u}(t; t_0, h_0) = \frac{U' \circ S(t; t_0, h_0)}{U^{\frac{\alpha}{2}} \circ S(t; t_0, h_0)}.$$

As S is continuous on $\mathcal{D}$ and u and $\dot{u}$ are the composition of continuous functions, they are continuous on $\mathcal{D}$. □

Next proposition will be very important. But before we state it we need two more lemmas necessary to carry on its proof.

Lemma 3.4. *Consider the autonomous equation* (12). *Suppose $P > 0$. Then*

$$\tau(h_0, P) < +\infty, \quad \forall\, h_0 < V_{max}.$$

Proof. Our aim now is to prove that the integral

$$\int_0^{u_{max}} \frac{du}{\sqrt{h_0 + Pu + \frac{1}{\alpha-1}\frac{1}{u^{\alpha-1}}}}$$

converges. Fix $h_0 < V_{max} < 0$. The function

$$\frac{1}{\sqrt{u^{\alpha-1}}\sqrt{h_0 + \frac{1}{\alpha-1}\frac{1}{u^{\alpha-1}}}} = \frac{1}{\sqrt{h_0 u^{\alpha-1} + \frac{1}{\alpha-1}}} \longrightarrow \sqrt{\alpha - 1}$$

as we let $u \to 0$. By definition of limit, this implies that

$$\forall\ \delta > 0\ \exists\ \varepsilon > 0: \quad u < \varepsilon \Rightarrow \frac{1}{\sqrt{h_0 + \frac{1}{\alpha-1}\frac{1}{u^{\alpha-1}}}} < \sqrt{u^{\alpha-1}}(\sqrt{\alpha-1}+\delta).$$

Given $\delta > 0$ define $C_\delta := \sqrt{\alpha-1}+\delta$ and fix an $\varepsilon > 0$ as above but sufficiently small so that

$$h_0 + \frac{1}{\alpha-1}\frac{1}{\varepsilon^{\alpha-1}} > 0.$$

Then

$$\int_0^\varepsilon \frac{du}{\sqrt{h_0 + Pu + \frac{1}{\alpha-1}\frac{1}{u^{\alpha-1}}}} \leqslant \int_0^\varepsilon \frac{du}{\sqrt{h_0 + \frac{1}{\alpha-1}\frac{1}{u^{\alpha-1}}}} < C_\delta \int_0^\varepsilon \sqrt{u^{\alpha-1}}\ du.$$

This integral is equal to a finite value and therefore it converges.

Now let us see that fixing some $\eta > 0$,

$$\int_{u_{max}-\eta}^{u_{max}} \frac{du}{\sqrt{h_0 + Pu + \frac{1}{\alpha-1}\frac{1}{u^{\alpha-1}}}} < +\infty.$$

Writing down $h_0 = -Pu_{max} - \frac{1}{\alpha-1}\frac{1}{u_{max}^{\alpha-1}}$ the integral becomes

$$\int \frac{du}{\sqrt{P(u-u_{max}) + \frac{1}{\alpha-1}\left(\frac{u_{max}^{\alpha-1}-u^{\alpha-1}}{u_{max}^{\alpha-1}u^{\alpha-1}}\right)}},$$

where we omit the extreme values of the integral for simplicity.

Dividing by the numerator, the function

$$\frac{\sqrt{u_{max}-u}}{\sqrt{P(u-u_{max}) + \frac{1}{\alpha-1}\left(\frac{u_{max}^{\alpha-1}-u^{\alpha-1}}{u_{max}^{\alpha-1}u^{\alpha-1}}\right)}}$$

equals

$$\frac{1}{\sqrt{-P + \frac{1}{\alpha-1}\left(\frac{u_{max}^{\alpha-1}-u^{\alpha-1}}{u_{max}^{\alpha-1}u^{\alpha-1}(u_{max}-u)}\right)}}.$$

This last function converges to $\frac{1}{\sqrt{-P+\frac{1}{u_{max}^{\alpha}}}} := l$, when we let $u \to u_{max}$ because

$$\frac{u_{max}^{\alpha-1}-u^{\alpha-1}}{u_{max}^{\alpha-1}u^{\alpha-1}(u_{max}-u)} \longrightarrow (\alpha-1)\frac{1}{u_{max}^{\alpha}}.$$

This can be seen by applying the Cauchy rule, or expanding $u^{\alpha-1}$ near $u_{max}^{\alpha-1}$. Notice that

$$\frac{1}{u_{max}^{\alpha}} - P > 0$$

once $u_{max} < P^{-\frac{1}{\alpha}}$.

By definition of convergence,

$$\forall\, \delta > 0 \ \ \exists\, \eta > 0 : \quad u_{max} - u < \eta \Rightarrow$$

$$\Rightarrow \frac{1}{\sqrt{P(u - u_{max}) + \frac{1}{\alpha-1}\left(\frac{u_{max}^{\alpha-1} - u^{\alpha-1}}{u_{max}^{\alpha-1} u^{\alpha-1}}\right)}} < \frac{1}{\sqrt{u_{max} - u}}(l + \delta).$$

Given $\delta > 0$ and defining $C_\delta := l + \delta$ we fix an η such as above. Then

$$\int_{u_{max}-\eta}^{u_{max}} \frac{du}{\sqrt{P(u - u_{max}) + \frac{1}{\alpha-1}\left(\frac{u_{max}^{\alpha-1} - u^{\alpha-1}}{u_{max}^{\alpha-1} u^{\alpha-1}}\right)}} < C_\delta \int_{u_{max}-\eta}^{u_{max}} \frac{du}{\sqrt{u_{max} - u}},$$

and this integral is convergent.

Therefore the integral

$$\int_0^{u_{max}} \frac{du}{\sqrt{h_0 + Pu + \frac{1}{\alpha-1}\frac{1}{u^{\alpha-1}}}}$$

converges because we proved converge in $]0, \varepsilon[$ and in $]u_{max} - \eta, u_{max}[$. In $[\varepsilon, u_{max} - \eta]$ the integrand is continuous and so the integral is convergent. □

Next lemma seems like a technical detail but it is essential to the study of equation (1), since it provides a bound for t_1.

Lemma 3.5. *Let u be a classical solution of equation (1) defined in $]t_0, t_1[$ and $\tau \in]t_0, t_1[$ such that*

$$u^\alpha(\tau)\|p\|_\infty < 1, \quad u(\tau) > 0, \quad \dot{u}(\tau) < 0.$$

Then for all $t \in]\tau, t_1[$ the following are satisfied:

1. $\ddot{u}(t) < 0$;
2. $\dot{u}(t) < \dot{u}(\tau)$;

Moreover

$$t_1 < \tau - \frac{u(\tau)}{\dot{u}(\tau)}.$$

Proof. Let $t_1 > \tau$ be the first instant of collision. Notice that

$$\ddot{u}(\tau) = -\frac{1}{u^\alpha(\tau)} + p(\tau) \leqslant -\frac{1}{u^\alpha(\tau)} + \|p\|_\infty < 0,$$

by the hypothesis.

We want to prove that $\ddot{u}(t) < 0$ and $\dot{u}(t) < \dot{u}(\tau)$, for every $t \in]\tau, t_1[$.

Suppose there was a first instant $t^* \in]\tau, t_1[$ such that

$$\dot{u}(t^*) = \dot{u}(\tau).$$

As t_1 is the first instant of collision we have that

$$u(t) > 0, \quad \forall\, t \in \ [\tau, t^*[$$

and

$$\dot{u}(t) < \dot{u}(\tau), \quad \forall\, t \in \ [\tau, t^*[.$$

Hence,

$$\dot{u}(t^*) = \dot{u}(\tau) + \int_\tau^{t^*} \left(-\frac{1}{u^\alpha(s)} + p(s) \right) \, ds.$$

Then

$$\int_\tau^{t^*} \left(-\frac{1}{u^\alpha(s)} + p(s) \right) \, ds = 0.$$

But $\dot{u}(s) < \dot{u}(\tau) < 0$, $\forall\, s \in \ [\tau, t^*[$, thus u is strictly decreasing in this interval. This implies

$$u^\alpha(s) < u^\alpha(\tau), \quad \forall\, s \in [\tau, t^*[$$

and consequently

$$-\frac{1}{u^\alpha(s)} < -\frac{1}{u^\alpha(\tau)}, \quad \forall\, s \in [\tau, t^*[.$$

$$\ddot{u}(s) = -\frac{1}{u^\alpha(s)} + p(s) \leqslant -\frac{1}{u^\alpha(\tau)} + \|p\|_\infty < 0,$$

for every $s \in [\tau, t^*[$. Hence,

$$\int_\tau^{t^*} \left(-\frac{1}{u^\alpha(s)} + p(s) \right) \, ds < 0,$$

which is a contradiction. Therefore $\dot{u}(t) < \dot{u}(\tau)$, for every $t \in]\tau, t_1[$. The same argument explained above can be used to conclude $\ddot{u}(t) < 0$ for every $t \in]\tau, t_1[$.

From here we can estimate a bound to t_1. For every $t \in]\tau, t_1[$,

$$u(t) - u(\tau) = \int_\tau^t \dot{u}(s) \, ds < \int_\tau^t \dot{u}(\tau) \, ds.$$

Thus,

$$0 < u(t) < u(\tau) + \dot{u}(\tau)(t - \tau), \ \forall\, t \in]\tau, t_1[.$$

Therefore, t_1 is finite and we can conclude that

$$t_1 < \tau - \frac{u(\tau)}{\dot{u}(\tau)}.$$

□

Proposition 3.6. *There exists a function $\psi : \mathbb{R} \longrightarrow \mathbb{R} \cup \{+\infty\}$ such that the domain of $\mathcal{P}$ is characterized by*

$$D = \{(t_0, h_0) \in \mathbb{R}^2 : \ h_0 < \psi(t_0)\}.$$

This function is 2π-periodic, lower semi-continuous and

$$\min_{\mathbb{R}} \psi \geqslant -\frac{\alpha}{\alpha - 1} \|p\|_\infty^{\frac{\alpha-1}{\alpha}}.$$

The successor map $\mathcal{P} : D \longrightarrow \mathbb{R}^2$, $\mathcal{P}(t_0, h_0) = (t_1, h_1)$ is injective and such that for each $t_0 \in \mathbb{R}$, the map

$$h_0 \in]-\infty, \psi(t_0)[\mapsto t_1(t_0, h_0)$$

is increasing, i.e., $\mathcal{P}$ is a twist map.

Proof. Recall that

$$D = \{(t_0, h_0) \in \mathbb{R}^2 : \ t_1 < +\infty\}.$$

If $(t_0, h_0) \in D$ then by Proposition 1.8.

$$\begin{cases} u(t_1; t_0, h_0) = 0 \\ \dot{u}(t_1; t_0, h_0) = -\infty. \end{cases}$$

In these conditions, we can find $\tau \in]t_0, t_1[$ satisfying the hypothesis of Lemma 3.5.

i) Let us prove that D is open.

As u and $\dot{u}$ are continuous in $\mathcal{D}$, we can choose an open neighbourhood B of (t_0, h_0) such that τ still satisfies the hypothesis of Lemma 3.5, i.e.,

$$u^\alpha(\tau; t_0', h_0')\|p\|_\infty < 1, \quad u(\tau; t_0', h_0') > 0, \quad \dot{u}(\tau; t_0', h_0') < 0,$$

for each $(t_0', h_0') \in B \subset \mathbb{R}^2$. Hence,

$$t_1' < \tau - \frac{u(\tau; t_0', h_0')}{\dot{u}(\tau; t_0', h_0')}$$

and so t_1' is finite and consequently $(t_0', h_0') \in D$.

Thus, $B \subseteq D$, and thus D is an open set.

ii) Let us now characterize the set D. Let $t_0 \in \mathbb{R}$ and consider the set

$$\mathcal{C}_{t_0} = \{h_0 \in \mathbb{R} : \ t_1 = t_1(t_0, h_0) < +\infty\}.$$

Let us compare the solutions of the equations

$$\ddot{u} = -\frac{1}{u^\alpha} + p(t) \tag{22}$$

$$\ddot{u} = -\frac{1}{u^\alpha} + \|p\|_\infty. \tag{23}$$

Recall that for the autonomous system we defined an integral of motion called the mechanical energy of the autonomous system

$$E = \frac{1}{2}\dot{u}^2(t) - \frac{1}{\alpha - 1}\frac{1}{u^{\alpha-1}} - Pu(t).$$

Define

$$V(t) = -\frac{1}{\alpha - 1}\frac{1}{u^{\alpha-1}} - Pu(t).$$

By Lemma 3.4, if

$$h_0 < -\frac{\alpha}{\alpha - 1}\|p\|_\infty^{\frac{\alpha-1}{\alpha}}$$

then the solution of equation (23) with collision in t_0 has a collision after a finite time.

Then by comparison of equations (22) and (23) and applying Lemma 2.4 we can deduce that

$$]-\infty, -\frac{\alpha}{\alpha - 1}\|p\|_\infty^{\frac{\alpha-1}{\alpha}}[\ \subseteq\ \mathcal{C}_{t_0}$$

and therefore $\mathcal{C}_{t_0}$ is not empty.

In fact we can deduce that $\mathcal{C}_{t_0}$ is an interval because, by Lemma 2.4, if $h^* \in \mathcal{C}_{t_0}$ then $]-\infty, h^*]\ \subseteq\ \mathcal{C}_{t_0}$.

Let $\psi(t_0) = \sup \mathcal{C}_{t_0}$. Observe that $\psi(t_0) \geqslant -\frac{\alpha}{\alpha-1}\|p\|_\infty^{\frac{\alpha-1}{\alpha}}$. It is now easy to conclude that

$$D = \{(t_0, h_0) \in \mathbb{R}^2 :\ h_0 < \psi(t_0)\}.$$

iii) The function ψ is lower semi-continuous.

Let $\alpha \in \mathbb{R}$ be such that $\alpha < \psi(t_0)$. By the characterization of D we have just proved $(t_0, \alpha) \in D$.

As D is an open set, there exists a ball with radius δ centered at (t_0, α), denoted by $B_\delta(t_0, \alpha)$, contained in D. In particular

$$]t_0 - \delta, t_0 + \delta[\times\{\alpha\} \subset B_\delta(t_0, \alpha) \subset D.$$

Then, by the characterization of D, $\alpha < \psi(t)$ for $t \in]t_0 - \delta, t_0 + \delta[$.

iv) Now let us prove that $\mathcal{P}$ is one-to-one. The bouncing solution satisfying the collision conditions $(t_0, h_0) \in \mathbb{R}^2$ is unique and is denoted by $u(\cdot; t_0, h_0)$.

Let $(t_0, h_0) \in D$. If $\mathcal{P}(t_0, h_0) = (t_1, h_1)$ then

$$u(\cdot; t_0, h_0) = u(\cdot; t_1, h_1)$$

by uniqueness. If

$$\mathcal{P}(t_0, h_0) = \mathcal{P}(\bar{t}_0, \bar{h}_0) = (t_1, h_1)$$

then
$$u(\cdot;t_0,h_0)=u(\cdot;t_1,h_1)=u(\cdot;\bar t_0,\bar h_0)$$
and the first instant of collision before t_1 is well defined and it is equal to $t_0=\bar t_0$ as well as the energy is $h_0=\bar h_0$.

v) Finally, let us prove that t_1 is increasing at t_0. Let us fix $t_0\in\mathbb{R}$ and
$$h_0<h_0^*<\psi(t_0).$$
By Lemma 2.4, we find that
$$t_1(t_0,h_0)<t_1(t_0,h_0^*).$$
Therefore the map
$$h_0\in\,]-\infty,\psi(t_0)[\;\mapsto\; t_1(t_0,h_0)$$
is increasing. □

3.2 The successor map is exact symplectic

Let $(t_0,h_0)\in\mathbb{R}^2$ and $\varepsilon>0$ such that $h_0+\frac{1}{\alpha-1}\frac{1}{\varepsilon^{\alpha-1}}>0$.

Let u be the solution of equation (1) with initial conditions
$$u(t_0)=\varepsilon,\quad \dot u(t_0)=\sqrt{2h_0+\frac{2}{\alpha-1}\frac{1}{\varepsilon^{\alpha-1}}}.$$

We will denote this solution by $u(t;t_0,h_0,\varepsilon)$. and set $u(t;t_0,h_0,0)=u(t;t_0,h_0)$.

Proposition 3.7. *Suppose $p(t)$ is of class C^1 and let $(t_0^*,h_0^*)\in D$. Then there exists $\varepsilon^*>0$, a neighbourhood V of (t_0^*,h_0^*) and two functions*
$$\tau,\ \mathcal{H}:V\times[0,\varepsilon^*]\longrightarrow\mathbb{R}$$
of class $C^{1,0}$ such that

1. *$\mathcal{P}(t_0,h_0)=(\tau(t_0,h_0,0),\mathcal{H}(t_0,h_0,0))$ for each $(t_0,h_0)\in V$.*
2. *Given $\varepsilon\in\,]0,\varepsilon^*]$, $\tau=\tau(t_0,h_0,\varepsilon)$ is such that*
   (a) $\tau>t_0$;
   (b) $\begin{cases}u(\tau;t_0,h_0,\varepsilon)=\varepsilon\\ u(t;t_0,h_0,\varepsilon)>\varepsilon,\ \forall t\in]t_0,\tau[.\end{cases}$
3. *For the same given ε, $\mathcal{H}=\mathcal{H}(t_0,h_0,\varepsilon)$ is such that*
$$\mathcal{H}=\frac{1}{2}\dot u(\tau;t_0,h_0,\varepsilon)^2-\frac{1}{\alpha-1}\frac{1}{\varepsilon^{\alpha-1}}.$$

Proof. Let $(t_0, h_0) \in \mathbb{R}^2$ and $\varepsilon > 0$ such that $h_0 + \frac{1}{\alpha-1}\frac{1}{\varepsilon^{\alpha-1}} > 0$.

We know there exists a unique solution of the regularized system satisfying the initial conditions

$$U(0) = \varepsilon, \quad V(0) = \sqrt{2h_0\varepsilon^\alpha + \frac{2\varepsilon}{\alpha - 1}}, \quad T(0) = t_0 \quad H(0) = h_0. \tag{24}$$

Let us denote the solution to this Cauchy problem by

$$(U(s; t_0, h_0, \varepsilon), V(s; t_0, h_0, \varepsilon), T(s; t_0, h_0, \varepsilon), H(s; t_0, h_0, \varepsilon)).$$

Observe that the integral of motion I of the regularized system introduced in Section 1.1 is such that $I(0) = 0$ over this solution. Therefore, it generates a solution of equation (1), by following the same steps as in that section.

Let us denote by $S(\cdot; t_0, h_0, \varepsilon)$ the inverse of $T(\cdot; t_0, h_0, \varepsilon)$.

i) Let us prove first that the solution

$$x(s; t_0, h_0, \varepsilon) = (U(s; t_0, h_0, \varepsilon), V(s; t_0, h_0, \varepsilon), T(s; t_0, h_0, \varepsilon), H(s; t_0, h_0, \varepsilon))$$

is of class $C^{1,0}$ in the variables (s, t_0, h_0) and ε.

The fact that x is continuous on ε is consequence of Theorem 5.14. The differentiability of x with respect to s, t_0 and h_0 and the continuity of the respective partial derivatives are consequence of Theorem 5.15.

ii) We will prove the existence of a $C^{1,0}$ function $s_1(t_0, h_0, \varepsilon)$ such that

$$T(s_1(t_0, h_0, 0); t_0, h_0, 0) = t_1.$$

Let us consider a point $(t_0^*, h_0^*) \in D$ and denote by t_1^* the first collision after t_0^*.

Let us take the solution of the regularized system $x(s; t_0^*, h_0^*, 0)$ generated by the solution $u(t; t_0^*, h_0^*, 0)$ of equation (1). As $S(t_1^*; t_0^*, h_0^*)$ is well defined, we denote

$$s_1^* = S(t_1^*; t_0^*, h_0^*).$$

Then

$$\begin{cases} U(s; t_0^*, h_0^*, 0) > 0 & \text{if } s \in]0, s_1^*[\\ U(s_1^*; t_0^*, h_0^*, 0) = 0. \end{cases}$$

Now we apply the modified version of the implicit function theorem stated in Lemma 5.4. to the system

$$\begin{cases} U'(s_1; t_0, h_0, \varepsilon) = -\sqrt{\frac{2\varepsilon}{\alpha-1} + 2\varepsilon^\alpha H(s_1; t_0, h_0, \varepsilon)} \\ s_1(t_0^*, h_0^*, 0) = s_1^*. \end{cases}$$

The function

$$F(s_1; t_0, h_0, \varepsilon) = U'(s_1; t_0, h_0, \varepsilon) + \sqrt{\frac{2\varepsilon}{\alpha - 1} + 2\varepsilon^\alpha H(s_1; t_0, h_0, \varepsilon)}$$

must satisfy the hypothesis of the theorem. In particular, we need to check that the derivative $\frac{dF}{ds_1}(s_1^*; t_0^*, h_0^*, 0) \neq 0$. We can see this using the definition of derivative

$$\frac{dF}{ds_1}(s_1^*; t_0^*, h_0^*, 0) = \lim_{r \to 0} \frac{F(s_1^* + r; t_0^*, h_0^*, 0) - F(s_1^*; t_0^*, h_0^*, 0)}{r}$$

$$= U''(s_1^*; t_0^*, h_0^*, 0) = \frac{1}{\alpha - 1}.$$

Hence, we deduce the existence of a function

$$s_1 = s_1(t_0, h_0, \varepsilon)$$

defined in a neighbourhood of $(t_0^*, h_0^*, 0)$ and differentiable with respect to t_0 and h_0, that satisfies the system.

iii) Let us prove now that this function s_1 is also solution of the system

$$\begin{cases} U(s_1(t_0, h_0, \varepsilon); t_0, h_0, \varepsilon) = \varepsilon \\ s_1(t_0^*, h_0^*, 0) = s_1^* \end{cases}$$

in a neighbourhood $V_1 \times [0, \varepsilon_1]$ of $(t_0^*, h_0^*, 0)$.

Let us fix $\nu > 0$ and $\eta > 0$ such that

$$\alpha(\alpha - 1)\eta^{\alpha - 1}\nu < 1$$

and

$$|h_1^*| < \nu$$

where h_1^* is the value of the energy function of the solution $u(t; t_0^*, h_0^*, 0)$ at the instant t_1^*.

Let us observe that

$$|H(s_1^*; t_0^*, h_0^*, 0)| = |h_1^*| < \nu,$$

$$|U(s_1^*; t_0^*, h_0^*, 0)| = 0 < \eta,$$

and that by continuous dependence we may choose $V_1 \times [0, \varepsilon_1]$ sufficiently small so that for $s_1 := s_1(t_0, h_0, \varepsilon)$ we have that

$$|H(s_1; t_0, h_0, \varepsilon)| \leqslant \nu, \quad |U(s_1; t_0, h_0, \varepsilon)| \leqslant \eta, \quad \text{for } (t_0, h_0, \varepsilon) \in V_1 \times [0, \varepsilon_1].$$

Consider the integral of motion associated to the solution $x(s;t_0,h_0,\varepsilon)$ of the regularized system. We know

$$I(s_1;t_0,h_0,\varepsilon)=0$$

as I is zero over the solutions of the system. Thus,

$$U'(s_1;t_0,h_0,\varepsilon)=-\sqrt{\frac{2}{\alpha-1}U(s_1;t_0,h_0,\varepsilon)+2U^\alpha(s_1;t_0,h_0,\varepsilon)H(s_1;t_0,h_0,\varepsilon)}.$$

Note that the map

$$f:[-\eta,\eta]\longrightarrow\mathbb{R}$$

defined by

$$f(\xi)=\frac{2}{\alpha-1}\xi+2\xi^\alpha H(s_1;t_0,h_0,\varepsilon)$$

is bijective. This is a consequence of our choice of neighbourhood. In fact we have

$$f'(\xi)=\frac{2}{\alpha-1}+2\alpha\xi^{\alpha-1}H(s_1;t_0,h_0,\varepsilon).$$

Hence, as $|H(s_1;t_0,h_0,\varepsilon)|<\nu$, $|\xi|<\eta$ and $\alpha(\alpha-1)\nu\eta^{\alpha-1}<1$ we get

$$f'(\xi)>0.$$

Now, as

1. by ii) we have
$$U'(s_1;t_0,h_0,\varepsilon)=-\sqrt{\frac{2\varepsilon}{\alpha-1}+2\varepsilon^\alpha H(s_1;t_0,h_0,\varepsilon)}.$$
Therefore $f(\varepsilon)=f(U(s_1;t_0,h_0,\varepsilon))$;
2. $U(s_1;t_0,h_0,\varepsilon)\in[-\eta,\eta]$;
3. f is bijective;

we conclude that

$$U(s_1;t_0,h_0,\varepsilon)=\varepsilon.$$

iv) We will prove now that

$$U(s;t_0,h_0,\varepsilon)>\varepsilon,\quad \forall\ s\in]0,s_1(t_0,h_0,\varepsilon)[,$$

$(t_0,h_0)\in V_2$ and $\varepsilon\in[0,\varepsilon_2]$, where $V_2\times[0,\varepsilon_2]\subseteq V_1\times[0,\varepsilon_1]$ is some neighbourhood of $(t_0^*,h_0^*,0)$.

Let us argue by contradiction. Suppose this is not true. Therefore there are sequences

$$\{(t_n,h_n)\}_{n\in\mathbb{N}},\quad (t_n,h_n)\to(t_0^*,h_0^*),$$

$$\{\varepsilon_n\}_{n\in\mathbb{N}}, \quad \varepsilon_n \searrow 0$$

and

$$\{\bar{s}_{1n}\}_{n\in\mathbb{N}}, \quad 0 < \bar{s}_{1n} < s_{1n} := s_1(t_n, h_n, \varepsilon_n)$$

such that

$$U(\bar{s}_{1n}; t_n, h_n, \varepsilon_n) = \varepsilon_n.$$

By continuity of s_1 we know that $s_{1n} \to s_1^*$.

Then $\{\bar{s}_{1n}\}$ is a bounded sequence and we can extract a convergent subsequence, denoted by $\{\bar{s}_{1k}\}$, with $\bar{s}_{1k} \to l$. Therefore $0 \leqslant l \leqslant s_1^*$.

As U is a continuous function of $(s; t_0, h_0, \varepsilon)$ we deduce

$$\begin{cases} U(\bar{s}_{1k}; t_k, h_k, \varepsilon_k) \to U(l; t_0^*, h_0^*, 0) \\ U(\bar{s}_{1k}; t_k, h_k, \varepsilon_k) = \varepsilon_k \to 0 \end{cases}$$

Hence, $U(l; t_0^*, h_0^*, 0) = 0$ and so $l = 0$ or $l = s_1^*$.

On the other hand U'' is also a continuous function of $(s; t_0, h_0, \varepsilon)$. Moreover,

$$\begin{cases} U''(0; t_0^*, h_0^*, 0) = \frac{1}{\alpha-1} \\ U''(s_1^*; t_0^*, h_0^*, 0) = \frac{1}{\alpha-1} \end{cases} \tag{25}$$

Thus, it is possible to find $k_0 > 0$ and $\delta > 0$ such that

$$U''(s; t_k, h_k, \varepsilon_k) \geqslant \frac{1}{\alpha}, \quad \forall\ s \in [0, \delta] \cup [s_{1k} - \delta, s_{1k}], \quad \forall\ k \geqslant k_0. \tag{26}$$

In particular, $U'(\cdot; t_k, h_k, \varepsilon_k)$ is strictly increasing on these intervals.

As

$$\begin{cases} U'(0, t_k, h_k, \varepsilon_k) = +\sqrt{2\varepsilon_k^\alpha h_k + \frac{2}{\alpha-1}\varepsilon_k} \\ U'(s_{1k}, t_k, h_k, \varepsilon_k) = -\sqrt{2\varepsilon_k^\alpha H(s_{1k}, t_k, h_k, \varepsilon_k) + \frac{2}{\alpha-1}\varepsilon_k} \end{cases} \tag{27}$$

then $U'(s_{1k}; t_k, h_k, \varepsilon_k) < 0$ and $U'(0; t_k, h_k, \varepsilon_k) > 0$.

Therefore, as a consequence of (26) and (27) we may conclude that

$$U(s; t_k, h_k, \varepsilon_k) > \varepsilon_k, \quad \forall\ s \in]0, \delta] \cup [s_{1k} - \delta, s_{1k}[. \tag{28}$$

Now, as $\bar{s}_{1k} \to 0$ or $\bar{s}_{1k} \to s_1^*$ for sufficiently large k, $\bar{s}_{1k} \in]0, \delta]$ or $\bar{s}_{1k} \in [s_1^* - \delta, s_1^*[$.

Also, by definition,

$$U(\bar{s}_{1k}; t_k, h_k, \varepsilon_k) = \varepsilon_k.$$

On the first interval we would get a contradiction. Thus, $\bar{s}_{1k}$ certainly belongs to $[s_1^* - \delta, s_1^*[$. However, notice that in this case $\bar{s}_{1k} \to s_1^*$. This means that $\bar{s}_{1k}$ is arbitrarily close to s_1^*. Simultaneously, the sequence s_{1k} is also arbitrarily close to s_1^*. Therefore, $\bar{s}_{1k}$ is arbitrarily close to s_{1k} and in particular there is a number N_0 such that

$$\bar{s}_{1k} \in]s_{1k} - \delta, s_{1k}[, \quad \forall\, k > N_0.$$

This contradicts (28) and the claim follows.

v) Finally we are ready to define the functions

$$\begin{cases} \tau(t_0, h_0, \varepsilon) = T(s_1(t_0, h_0, \varepsilon); t_0, h_0, \varepsilon) \\ \mathcal{H}(t_0, h_0, \varepsilon) = H(s_1(t_0, h_0, \varepsilon); t_0, h_0, \varepsilon). \end{cases}$$

Moreover this functions

- are of class $C^{1,0}$;
- verify properties *1.*,*2.* and *3.* of the proposition.

Let us check these claims:

- differentiability with respect to t_0 and h_0 are consequence of the chain rule and of the differentiability of the function s_1. Continuity on ε is consequence of continuous dependence on parameters and continuity of s_1.
- *1.*
$$\begin{cases} \tau(t_0, h_0, 0) = T(s_1(t_0, h_0, 0); t_0, h_0, 0) = t_1 \\ \mathcal{H}(t_0, h_0, 0) = H(s_1(t_0, h_0, 0); t_0, h_0, 0) = h_1 \end{cases}$$

  2. (a) Let $t_0 = T(0; t_0, h_0, \varepsilon)$ and $\tau = T(s_1; t_0, h_0, \varepsilon)$ with $s_1 > 0$. Then as

  $$T' = U^{\frac{\alpha}{2}}$$

  $T'(s_1; t_0, h_0, \varepsilon) > 0$ at least for some neighbourhood of $\varepsilon = 0$. Therefore, $\tau > t_0$.

  (b) Notice

  $$u(\tau; t_0, h_0, \varepsilon) = U(S(\tau; t_0, h_0, \varepsilon); t_0, h_0, \varepsilon) = U(s_1; t_0, h_0, \varepsilon) = \varepsilon.$$

  Simultaneously, for each $t \in]t_0, \tau[$ we have $S(t; t_0, h_0, \varepsilon) \in]0, s_1[$. Thus

  $$u(t; t_0, h_0, \varepsilon) = U(S(t; t_0, h_0, \varepsilon); t_0, h_0, \varepsilon) > \varepsilon.$$

3. Observing that $I(S(\tau; t_0, h_0, \varepsilon); t_0, h_0, \varepsilon) = 0$ and that

$$U'^2(S(t; t_0, h_0, \varepsilon)) = \dot{u}(t)^2 U^\alpha(S(t; t_0, h_0, \varepsilon))$$

We immediately conclude that

$$\mathcal{H}(t_0, h_0, \varepsilon) = H(S(\tau; t_0, h_0, \varepsilon); t_0, h_0, \varepsilon) = \frac{1}{2}\dot{u}^2(\tau; t_0, h_0, \varepsilon) - \frac{1}{\alpha - 1}\frac{1}{\varepsilon^{\alpha-1}}$$

□

From now on, let $\varphi : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ such that

$$\varphi(t_0, h_0) = (\cos t_0, \sin t_0, h_0)$$

be a parametrization of the cylinder.

Let Γ be a curve in the cylinder and denote by γ a curve in the plane such that $\varphi \circ \gamma = \Gamma$. Γ is the image of γ by φ. We will denote $\gamma_1 := \mathcal{P} \circ \gamma$ and

$$\Gamma_1 := \varphi \circ \gamma_1.$$

Therefore defining $\overline{\mathcal{P}} := \varphi \circ \mathcal{P} \circ \varphi^{-1}$, we may say that

$$\Gamma_1 = \overline{\mathcal{P}} \circ \Gamma.$$

Before we start with the last proposition of this section we need one more lemma.

Lemma 3.8. *If for each Jordan curve on the cylinder* Γ

$$\int_\Gamma \omega = \int_{\Gamma_1} \omega,$$

where $\omega = h_0 dt_0$ *and* $\Gamma_1 = \overline{\mathcal{P}}(\Gamma)$, *then the differential form*

$$h_1 dt_1 - h_0 dt_0,$$

is exact on the cylinder.

Proof. By Proposition 3.7., the map

$$\mathcal{P}(t_0, h_0) = (\ \tau(t_0, h_0, 0), \mathcal{H}(t_0, h_0, 0)\ )$$

is differentiable. Also by Proposition 3.6., $\mathcal{P}$ is injective and therefore it is bijective onto its image. Moreover, we can invert the map $\mathcal{P}$ because by reverting time we can determine the initial collision conditions given some final collision conditions $(t_1, h_1) \in \mathcal{P}(D)$. Analogously, this inverse map is a successor map and so it is differentiable. Then

$$\mathcal{P} : D \longrightarrow \mathcal{P}(D)$$

is a diffeomorphism between the two sets. Therefore, it follows by Theorem 5.6. that

$$\int_\Gamma h_1\ dt_1 = \int_\Gamma h_0 \circ \mathcal{P}\ d(t_0 \circ \mathcal{P}) = \int_\Gamma \mathcal{P}^*(h_0\ dt_0) = \int_{\mathcal{P}(\Gamma)} h_0\ dt_0 = \int_{\Gamma_1} \omega.$$

Using Theorem 5.7., we define in D the pullback orientation induced by $\mathcal{P}$ which, by definition, makes $\mathcal{P}$ an orientation-preserving map.

Hence, we proved that

$$\int_\Gamma h_0\ dt_0 = \int_\Gamma h_1\ dt_1,$$

and so

$$\int_\Gamma h_0\ dt_0 - h_1\ dt_1 = 0$$

By Theorem 5.5., a 1-form on a differentiable manifold is conservative if and only if it is exact on it. Thus $h_0\ dt_0 - h_1\ dt_1$ is exact on the cylinder. □

Proposition 3.9. *Suppose p is of class C^1. Then the differential form*

$$h_1 dt_1 - h_0 dt_0,$$

where $(t_1, h_1) = \mathcal{P}(t_0, h_0)$, is exact in the cylinder. That is to say, there exists a function $G \in C^1(D)$, $G = G(t_0, h_0)$ 2π-periodic on the variable t_0 such that

$$dG = h_1 dt_1 - h_0 dt_0.$$

Proof. Consider the differential form

$$\omega = h_0\ dt_0$$

in the cylinder.

Our goal is to prove that

$$\int_\Gamma \omega = \int_{\Gamma_1} \omega$$

for every smooth Jordan curve Γ on the cylinder contained in the image of $D = \{(t_0, h_0) \in \mathbb{R}^2 | h_0 < \psi(t_0)\}$ by φ and

$$\Gamma_1 = \overline{\mathcal{P}}(\Gamma).$$

Let us take a curve Γ in the cylinder such as this and let γ be such as before and defined in the whole $\mathbb{R}$. Then $\gamma : \mathbb{R} \longrightarrow \mathbb{R}^2$ is such that

1. $\gamma(\mathbb{R}) \subset D$.
2. $\varphi \circ \mathcal{P} \circ \gamma(\mathbb{R}) = \Gamma_1$.

Let us suppose that Γ is not contractible to a point. The contractible case follows analogously.

Denote $\gamma(s) = (t_0(s), h_0(s))$, $s \in \mathbb{R}$. And take Γ such that

$$\begin{cases} t_0(s+2\pi) = t_0(s) + 2\pi \\ h_0(s+2\pi) = h_0(s). \end{cases}$$

for every $s \in \mathbb{R}$.

Take $(t_0^*, h_0^*) \in \gamma(\mathbb{R})$ with $\gamma(s^*) = (t_0^*, h_0^*)$. Then $\gamma([s^*, s^*+2\pi]) \subset D$ is a bounded and closed set in $\mathbb{R}^2$. Therefore it is a compact subset.

By Proposition 3.7., for each $(t_0(s), h_0(s)) \in \gamma([s^*, s^*+2\pi])$ there exists a neighbourhood $V(s) \subseteq \mathbb{R}^2$ of $(t_0(s), h_0(s))$ and a number $\varepsilon(s) > 0$ where the statements of this Proposition are verified.

By compactness, there exists a finite number of points $s_1, \cdots, s_N$ such that

$$\gamma([s^*, s^*+2\pi]) \subset \bigcup_{i=1}^{N} V(s_i) := V.$$

Also take $\varepsilon_1 := \min\{\varepsilon(s_1), \cdots, \varepsilon(s_N)\}$.

In $V \times [0, \varepsilon_1]$ the functions $\tau, \mathcal{H}$ described in Proposition 3.7. are well defined.

Consider now the coordinates (u, v, t) in $\mathbb{R}^3$ and the surface M_ε with parametric equations

$$\begin{cases} u = u(t'; t_0(s), h_0(s), \varepsilon) \\ v = \dot{u}(t'; t_0(s), h_0(s), \varepsilon) \\ t = t' \end{cases} \tag{29}$$

with parameters $s \in [0, 2\pi]$ and $t' \in [t_0(s), \tau(t_0(s), h_0(s), \varepsilon)]$. Notice that the surface M_ε is contained in the space $\{u \geqslant \varepsilon\}$.

More importantly notice that M_ε is a surface with four corners. Let us check that it is regular outside its corners. That is, we have to check that the map (29), which we will denote by $X(s, t')$ is a parametrization of M_ε. This means we have to check that X is bijective onto its image, of class C^1 and the Jacobian matrix $DX_{(s,t')}$ has rank 2.

1. Injective:

   Let (s_1, t_1') and $(s_2, t_2') \in \mathbb{R}^2$ be such that

   $$X(s_1, t_1') = X(s_2, t_2').$$

   In one hand it is immediate that $t_1' = t_2' := t'$. On the other we get that

   $$\begin{cases} u(t'; t_0(s_1), h_0(s_1), \varepsilon) = u(t'; t_0(s_2), h_0(s_2), \varepsilon) \\ \dot{u}(t'; t_0(s_1), h_0(s_1), \varepsilon) = \dot{u}(t'; t_0(s_2), h_0(s_2), \varepsilon) \end{cases}$$

As u is solution of an equation for which there is uniqueness, both solutions are equal. In particular

$$\begin{cases} t_0(s_1) = t_0(s_2) \\ h_0(s_1) = h_0(s_2) \end{cases}$$

As γ is a Jordan curve in the plane and we are taking $s \in [0, 2\pi]$ it is thereby injective restricted to this interval. Therefore, $s_1 = s_2$.

2. We can see that X is of class C^1 by consequence of the chain rule, as $u \in C^2$ and γ is differentiable (in fact, it is smooth).

3. $DX_{(s,t')}$ has rank 2:

   Let $X(s, t')$ be the map

$$\begin{cases} u(s, t') = u(t'; t_0(s), h_0(s), \varepsilon) \\ v(s, t') = \dot{u}(t'; t_0(s), h_0(s), \varepsilon) \\ t(s, t') = t'. \end{cases}$$

   Then we can write

$$DX_{(s,t')} = \begin{bmatrix} \frac{\partial u}{\partial s}(s,t') & \frac{\partial u}{\partial t'}(s,t') \\ \\ \frac{\partial v}{\partial s}(s,t') & \frac{\partial v}{\partial t'}(s,t') \\ \\ \frac{\partial t}{\partial s}(s,t') & \frac{\partial t}{\partial t'}(s,t') \end{bmatrix}$$

$$DX_{(s,t')} = \begin{bmatrix} \frac{\partial u}{\partial t_0}(s,t')t_0'(s) + \frac{\partial u}{\partial h_0}(s,t')h_0'(s) & \dot{u}(s,t') \\ \\ \frac{\partial \dot{u}}{\partial t_0}(s,t')t_0'(s) + \frac{\partial \dot{u}}{\partial h_0}(s,t')h_0'(s) & \ddot{u}(s,t') \\ \\ 0 & 1 \end{bmatrix}$$

   In order to compute the derivatives with respect to t_0 and h_0 we will use Theorem 5.15.

   Let $y = (u, v)$ be such that $\dot{y} = Y(t, y)$ with

$$\begin{cases} \dot{u} = v \\ \dot{v} = -\frac{1}{u^\alpha} + p(t) \end{cases} \tag{30}$$

   and

$$\frac{\partial Y}{\partial y}(t, y) = \begin{pmatrix} 0 & 1 \\ \frac{\alpha}{u^{\alpha+1}} & 0 \end{pmatrix}.$$

   $Y(t, y)$ is well defined in $\mathbb{R} \times]0, +\infty[\times \mathbb{R}$. Then by Theorem 5.15. the general classical solution $y(t; t_0, y_0)$ is differentiable in the three variables. Moreover we know how to compute its derivatives:

(a) $\frac{\partial y}{\partial t_0}$ is the solution of the problem

$$\begin{cases} \dot{x} = \frac{\partial Y}{\partial y} x \\ x(t_0) = -Y(t_0, y_0) \end{cases}$$

Denoting $x = (x_1, x_2)$ and $y_0 = (u_0, v_0)$ we may rewrite this system as

$$\begin{cases} \dot{x_1} = x_2 \\ \dot{x_2} = \frac{\alpha}{u^{\alpha+1}} x_1 \\ x(t_0) = -(v_0, -\frac{1}{u_0^\alpha} + p(t_0)). \end{cases}$$

Hence, as $\frac{\partial y}{\partial t_0} = \left(\frac{\partial u}{\partial t_0}, \frac{\partial v}{\partial t_0} \right)$ is the solution of this system, we find that

$$\begin{cases} \frac{d}{dt}\left(\frac{\partial u}{\partial t_0}\right) = \frac{\partial v}{\partial t_0} \\ \frac{d}{dt}\left(\frac{\partial v}{\partial t_0}\right) = \frac{\alpha}{u^{\alpha+1}} \frac{\partial u}{\partial t_0} \\ \frac{\partial u}{\partial t_0}(t_0) = -v_0 \\ \frac{\partial v}{\partial t_0}(t_0) = \frac{1}{u_0^\alpha} - p(t_0) \end{cases} \tag{31}$$

From the upper two equalities we deduce

$$\frac{d^2}{dt^2}\left(\frac{\partial u}{\partial t_0}\right) = \frac{\alpha}{u^{\alpha+1}(t; t_0, y_0)} \frac{\partial u}{\partial t_0} \tag{32}$$

(b) $\frac{\partial y}{\partial y_0}$ is the solution of the problem

$$\begin{cases} \dot{X} = \frac{\partial Y}{\partial y} X \\ X(t_0) = Id \end{cases}$$

where

$$\frac{\partial y}{\partial y_0} = \begin{bmatrix} \frac{\partial u}{\partial u_0} & \frac{\partial u}{\partial v_0} \\ \\ \frac{\partial v}{\partial u_0} & \frac{\partial v}{\partial v_0} \end{bmatrix}.$$

Denoting the matrix X by

$$X = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix},$$

we can rewrite the system as

$$\begin{cases} \dot{x}_{11} = x_{21} \\ \dot{x}_{12} = x_{22} \\ \dot{x}_{21} = \frac{\alpha}{u^{\alpha+1}} x_{11} \\ \dot{x}_{22} = \frac{\alpha}{u^{\alpha+1}} x_{12} \\ X(t_0) = Id. \end{cases}$$

As $\frac{\partial y}{\partial y_0}$ is the solution of this system we find that

$$\begin{cases} \frac{d}{dt}\left(\frac{\partial u}{\partial u_0}\right) = \frac{\partial v}{\partial u_0} \\ \frac{d}{dt}\left(\frac{\partial u}{\partial v_0}\right) = \frac{\partial v}{\partial v_0} \end{cases}$$

and

$$\begin{cases} \frac{d}{dt}\left(\frac{\partial v}{\partial u_0}\right) = \frac{\alpha}{u^{\alpha+1}}\frac{\partial u}{\partial u_0} \\ \frac{d}{dt}\left(\frac{\partial v}{\partial v_0}\right) = \frac{\alpha}{u^{\alpha+1}}\frac{\partial u}{\partial v_0} \end{cases}$$

subject to the initial conditions

$$\begin{cases} \frac{\partial u}{\partial u_0}(t_0) = 1 \\ \frac{\partial u}{\partial v_0}(t_0) = 0 \\ \frac{\partial v}{\partial u_0}(t_0) = 0 \\ \frac{\partial v}{\partial v_0}(t_0) = 1. \end{cases} \tag{33}$$

Therefore, as in the last case we may deduce the following equalities

$$\frac{d^2}{dt^2}\left(\frac{\partial u}{\partial u_0}\right) = \frac{\alpha}{u^{\alpha+1}}\frac{\partial u}{\partial u_0} \tag{34}$$

$$\frac{d^2}{dt^2}\left(\frac{\partial u}{\partial v_0}\right) = \frac{\alpha}{u^{\alpha+1}}\frac{\partial u}{\partial v_0} \tag{35}$$

Let $u(t; t_0(s), h_0(s), \varepsilon)$ be the solution of system (30) with initial conditions

$$t_0 = t_0(s), \quad u_0 = \varepsilon, \quad v_0 = \sqrt{2h_0 + \frac{2}{\alpha-1}\frac{1}{\varepsilon^{\alpha-1}}}.$$

Then, by (31) and (32), $\frac{\partial u}{\partial t_0}$ satisfies

$$\frac{d^2}{dt^2}\left(\frac{\partial u}{\partial t_0}\right) = \frac{\alpha}{u^{\alpha+1}(t; t_0(s), h_0(s), \varepsilon)}\frac{\partial u}{\partial t_0},$$

$$\frac{\partial u}{\partial t_0}(t_0) = -\sqrt{2h_0 + \frac{2}{\alpha-1}\frac{1}{\varepsilon^{\alpha-1}}}, \quad \frac{d}{dt}\left(\frac{\partial u}{\partial t_0}\right)(t_0) = \frac{1}{\varepsilon^{\alpha}} - p(t_0).$$

Also, by (33) and (35) $\frac{\partial u}{\partial v_0}$ satisfies

$$\frac{d^2}{dt^2}\left(\frac{\partial u}{\partial v_0}\right) = \frac{\alpha}{u^{\alpha+1}(t; t_0(s), h_0(s), \varepsilon)}\frac{\partial u}{\partial v_0},$$

$$\frac{\partial u}{\partial v_0}(t_0) = 0, \quad \frac{d}{dt}\left(\frac{\partial u}{\partial v_0}\right)(t_0) = 1.$$

Now, notice

$$\frac{\partial u}{\partial h_0} = \frac{\partial u}{\partial v_0}\frac{\partial v_0}{\partial h_0} = \frac{1}{v_0}\frac{\partial u}{\partial v_0}.$$

This allows us to compute the following equalities for $\frac{\partial u}{\partial h_0}$:

$$\frac{d^2}{dt^2}\left(\frac{\partial u}{\partial h_0}\right) = \frac{\alpha}{u^{\alpha+1}(t;t_0(s),h_0(s),\varepsilon)}\frac{\partial u}{\partial h_0},$$

$$\frac{\partial u}{\partial h_0}(t_0) = 0, \quad \frac{d}{dt}\left(\frac{\partial u}{\partial h_0}\right)(t_0) = \frac{1}{v_0} = \frac{1}{\sqrt{2h_0 + \frac{2}{\alpha-1}\frac{1}{\varepsilon^{\alpha-1}}}}.$$

So we have found that $y_1 = \frac{\partial u}{\partial t_0}$ and $y_2 = \frac{\partial u}{\partial h_0}$ are both solutions of the differential equation

$$\ddot{y} = \frac{\alpha}{u^{\alpha+1}(t;t_0(s),h_0(s),\varepsilon)}y$$

with initial conditions $y_1(t_0) = -\sqrt{2h_0 + \frac{2}{\alpha-1}\frac{1}{\varepsilon^{\alpha-1}}}$ and $\dot{y}_1(t_0) = \frac{1}{\varepsilon^\alpha} - p(t_0)$; $y_2(t_0) = 0$ and $\dot{y}_2(t_0) = \left(2h_0 + \frac{2}{\alpha-1}\frac{1}{\varepsilon^{\alpha-1}}\right)^{-\frac{1}{2}}$.

Defining $z_1 = \frac{\partial v}{\partial t_0}$ and $z_2 = \frac{\partial v}{\partial h_0}$ we find that (y_1, z_1) and (y_2, z_2) are both solutions of the system

$$\begin{cases} \dot{y} = z \\ \dot{z} = \frac{\alpha}{u^{\alpha+1}}y \end{cases}$$

which is associated to the vector field

$$A(t) = \begin{pmatrix} 0 & 1 \\ \frac{\alpha}{u^{\alpha+1}} & 0 \end{pmatrix}.$$

Then, applying Jacobi-Liouville's formula,

$$\det\begin{pmatrix} \frac{\partial u}{\partial t_0} & \frac{\partial u}{\partial h_0} \\ \frac{\partial v}{\partial t_0} & \frac{\partial v}{\partial h_0} \end{pmatrix} = \det\begin{pmatrix} \frac{\partial u}{\partial t_0}(t_0) & \frac{\partial u}{\partial h_0}(t_0) \\ \frac{\partial v}{\partial t_0}(t_0) & \frac{\partial v}{\partial h_0}(t_0) \end{pmatrix} e^{\int_{t_0}^{t}\mathrm{tr}(A(s))\ ds},$$

as $\mathrm{tr}(A(s)) \equiv 0$ and as $\frac{\partial u}{\partial t_0}(t_0)\frac{\partial v}{\partial h_0}(t_0) - \frac{\partial u}{\partial h_0}(t_0)\frac{\partial v}{\partial t_0}(t_0) = -1$, we get

$$\det\begin{pmatrix} \frac{\partial u}{\partial t_0} & \frac{\partial u}{\partial h_0} \\ \frac{\partial v}{\partial t_0} & \frac{\partial v}{\partial h_0} \end{pmatrix} = -1.$$

Moreover, as γ is smooth

$$\gamma'(s) = (t_0'(s), h_0'(s))$$

is nowhere zero.

Getting back to

$$DX_{(s,t')} = \begin{bmatrix} \frac{\partial u}{\partial t_0}(s,t')t_0'(s) + \frac{\partial u}{\partial h_0}(s,t')h_0'(s) & \dot{u}(s,t') \\ \\ \frac{\partial \dot{u}}{\partial t_0}(s,t')t_0'(s) + \frac{\partial \dot{u}}{\partial h_0}(s,t')h_0'(s) & \ddot{u}(s,t') \\ \\ 0 & 1 \end{bmatrix}$$

It is easy to see that

$$\det \begin{pmatrix} \frac{\partial \dot{u}}{\partial t_0}(s,t')t_0'(s) + \frac{\partial \dot{u}}{\partial h_0}(s,t')h_0'(s) & \ddot{u}(s,t') \\ \\ 0 & 1 \end{pmatrix}$$

and

$$\det \begin{pmatrix} \frac{\partial u}{\partial t_0}(s,t')t_0'(s) + \frac{\partial u}{\partial h_0}(s,t')h_0'(s) & \dot{u}(s,t') \\ 0 & 1 \end{pmatrix}$$

cannot be simultaneously zero, due to our previous conclusions.

Hence, $DX_{(s,t')}$ has rank 2.

Now we are going to apply Stokes' Theorem on the surface M_ε. Let us consider the *Poincaré-Cartan differential form*

$$\Omega = v\ du - E(u,v,t)\ dt,$$

where $E(u,v,t) = \frac{1}{2}v^2 - \frac{1}{\alpha-1}\frac{1}{u^{\alpha-1}} - p(t)u$.

Let us restrict $\Omega(u,v,t)$ to M_ε using the pullback by the inclusion of M_ε into $\mathbb{R}^3$. Let Σ be this restriction. Then

$$\Sigma = i^*\Omega$$

where $i : M_\varepsilon \hookrightarrow \mathbb{R}^3$.

Therefore,

$$\Sigma = (v \circ i)\ d(u \circ i) - E \circ i\ d(t \circ i)$$

$$\Sigma = v(t';t_0(s),h_0(s),\varepsilon)\ du - E(u(t';t_0(s),h_0(s),\varepsilon), v(t';t_0(s),h_0(s),\varepsilon), t')\ dt'.$$

Note that in the last line du is the differential of the function $u(t';t_0(s),h_0(s),\varepsilon)$. By simplicity we will now omit the functions arguments. So we may write

$$\Sigma = v\ \left(\frac{\partial u}{\partial s}ds + \frac{\partial u}{\partial t'}dt'\right) - E\ dt'$$

and so pointing out that $\frac{\partial u}{\partial t'} = v$ we get

$$\Sigma = (v^2 - E)dt' + vu_s ds.$$

After some direct computations, we get

$$d\Sigma = (vv_s - \ddot{u}u_s - E_u u_s - E_v v_s) ds \wedge dt'.$$

Notice that $E_v = v$ and

$$E_u(u, v, t') = \frac{1}{u^\alpha} - p(t) = -\ddot{u}.$$

Therefore $d\Sigma = 0$.

Hence applying Stokes' Theorem

$$\int_{\partial M_\varepsilon} \Sigma = \int_{M_\varepsilon} d\Sigma = 0.$$

The boundary is the union of the following curves

$$\partial M_\varepsilon = \Gamma_\varepsilon \cup \bar{\gamma}_\varepsilon \cup \bar{\Gamma}_\varepsilon \cup \gamma_\varepsilon,$$

where

1. Γ_ε is the curve obtained by taking $t' = t_0(s)$. This is

$$\Gamma_\varepsilon(s) = \left(\varepsilon, \sqrt{2h_0(s) + \frac{2}{\alpha - 1}\frac{1}{\varepsilon^{\alpha-1}}}, t_0(s)\right).$$

2. $\bar{\Gamma}_\varepsilon$ is the curve defined by taking $t' = \tau(t_0(s), h_0(s), \varepsilon)$. This holds

$$\bar{\Gamma}_\varepsilon(s) = \left(\varepsilon, -\sqrt{2\mathcal{H}(t_0(s), h_0(s), \varepsilon) + \frac{2}{\alpha - 1}\frac{1}{\varepsilon^{\alpha-1}}}, \tau(t_0(s), h_0(s), \varepsilon)\right).$$

3. $\gamma_\varepsilon(t')$ is the curve associated to the trajectory of u and $\dot{u}$ with initial conditions $(t_0(0), h_0(0))$, obtained by taking $s = 0$.

4. $\bar{\gamma}_\varepsilon(t')$ is the curve associated to the trajectory of u and $\dot{u}$ with initial conditions $(t_0(2\pi), h_0(2\pi))$, obtained by taking $s = 2\pi$.

To compute the integral over the boundary of M_ε we must choose an orientation for the boundary.

We will follow the curve Γ_ε starting in the point with parameter $s = 0$ and finishing in $s = 2\pi$. Notice $\Gamma_\varepsilon(2\pi) = \bar{\gamma}_\varepsilon(t_0(2\pi))$ and so from here we can walk $\bar{\gamma}_\varepsilon$ from $t_0(2\pi)$ to $\tau(t_0(2\pi), h_0(2\pi), \varepsilon)$. Next we walk $\bar{\Gamma}_\varepsilon$ on the opposite direction from $s = 2\pi$ to $s = 0$. And finally we walk γ_ε from $t' = \tau(t_0(0), h_0(0), \varepsilon)$ to $t' = t_0(0)$.

This implies that

$$\int_{\partial M_\varepsilon} = \int_{\Gamma_\varepsilon} + \int_{\bar{\gamma}_\varepsilon} - \int_{\bar{\Gamma}_\varepsilon} - \int_{\gamma_\varepsilon}.$$

By Proposition 3.1.,

$$u(t;t_0(2\pi),h_0(2\pi),\varepsilon)=u(t;t_0(0)+2\pi,h_0(0),\varepsilon)=u(t-2\pi;t_0(0),h_0(0),\varepsilon).$$

Therefore, $\bar{\gamma}_\varepsilon(t'+2\pi)=\gamma_\varepsilon(t')+(0,0,2\pi)$.
Let us prove that

$$\int_{\bar{\gamma}_\varepsilon}\Sigma=\int_{\gamma_\varepsilon}\Sigma.$$

$$\int_{\bar{\gamma}_\varepsilon}\Sigma=\int_{t_0(2\pi)}^{\tau(t_0(2\pi),h_0(2\pi),\varepsilon)}\bar{\gamma}_\varepsilon^*\Sigma=\int_{t_0(2\pi)}^{\tau(t_0(2\pi),h_0(2\pi),\varepsilon)}(\dot{u}^2-E)d(t'\circ\bar{\gamma}_\varepsilon)+\dot{u}u_s d(s\circ\bar{\gamma}_\varepsilon)$$

Note that

$$d(t'\circ\bar{\gamma}_\varepsilon)=\frac{\partial}{\partial t'}t'\ dt'=dt',$$

$$d(s\circ\bar{\gamma}_\varepsilon)=\frac{\partial}{\partial t'}s\ dt'=0$$

where $(s,t')=X^{-1}\circ\bar{\gamma}_\varepsilon(t')$.

Therefore,

$$\int_{\bar{\gamma}_\varepsilon}\Sigma=\int_{t_0(2\pi)}^{\tau(t_0(2\pi),h_0(2\pi),\varepsilon)}(\dot{u}^2-E)\circ\bar{\gamma}_\varepsilon\ dt'$$

By definition of $\bar{\gamma}_\varepsilon$, we know that

$$(\dot{u}^2-E)\circ\bar{\gamma}_\varepsilon=\dot{u}^2(t';t_0(2\pi),h_0(2\pi),\varepsilon)-E(t';t_0(2\pi),h_0(2\pi),\varepsilon)$$

where

$$E(t';t_0(2\pi),h_0(2\pi),\varepsilon)=E(u(t';t_0(2\pi),h_0(2\pi),\varepsilon),\dot{u}(t';t_0(2\pi),h_0(2\pi),\varepsilon),t').$$

Then

$$\int_{\bar{\gamma}_\varepsilon}\Sigma=\int_{t_0(2\pi)}^{\tau(t_0(2\pi),h_0(2\pi),\varepsilon)}\dot{u}^2(t'-2\pi;t_0(0),h_0(0),\varepsilon)-E(t'-2\pi;t_0(0),h_0(0),\varepsilon)dt'.$$

Now we operate a change of variables. Let

$$s=t'-2\pi.$$

Then

$$\int_{\bar{\gamma}_\varepsilon}\Sigma=\int_{t_0(0)}^{\tau(t_0(0),h_(0),\varepsilon)}\dot{u}^2(s;t_0(0),h_0(0),\varepsilon)-E(s;t_0(0),h_0(0),\varepsilon)ds.$$

Hence, by definition of γ_ε

$$\int_{\bar{\gamma}_\varepsilon}\Sigma=\int_{t_0(0)}^{\tau(t_0(0),h_(0),\varepsilon)}(\dot{u}^2-E)\circ\gamma_\varepsilon(t')\ dt'=\int_{\gamma_\varepsilon}\Sigma.$$

As

$$0 = \int_{\partial M_\varepsilon} \Sigma = \int_{\Gamma_\varepsilon} \Sigma \ + \int_{\bar{\gamma}_\varepsilon} \Sigma \ - \int_{\bar{\Gamma}_\varepsilon} \Sigma \ - \ \Sigma \int_{\gamma_\varepsilon},$$

we conclude

$$\int_{\Gamma_\varepsilon} \Sigma = \int_{\bar{\Gamma}_\varepsilon} \Sigma.$$

Notice that

$$\Gamma_\varepsilon^* \Sigma = \Gamma_\varepsilon^* i^* \Omega = \Gamma_\varepsilon^* \Omega,$$

thus

$$\int_{\Gamma_\varepsilon} \Sigma = \int_0^{2\pi} \Gamma_\varepsilon^* \Omega = \int_0^{2\pi} v \circ \Gamma_\varepsilon d(u \circ \Gamma_\varepsilon) - E \circ \Gamma_\varepsilon d(t \circ \Gamma_\varepsilon),$$

and analogously

$$\int_{\bar{\Gamma}_\varepsilon} \Sigma = \int_0^{2\pi} v \circ \bar{\Gamma}_\varepsilon d(u \circ \bar{\Gamma}_\varepsilon) - E \circ \bar{\Gamma}_\varepsilon d(t \circ \bar{\Gamma}_\varepsilon).$$

By definition of Γ_ε and $\bar{\Gamma}_\varepsilon$, in both cases

$$u \equiv \varepsilon.$$

Therefore, we get the equality

$$\int_0^{2\pi} E \circ \Gamma_\varepsilon \ d(t \circ \Gamma_\varepsilon) = \int_0^{2\pi} E \circ \bar{\Gamma}_\varepsilon \ d(t \circ \bar{\Gamma}_\varepsilon).$$

Let us look closer into these expressions:

$$\int_0^{2\pi} E \circ \Gamma_\varepsilon \ d(t \circ \Gamma_\varepsilon) = \int_0^{2\pi} [\frac{1}{2}(\dot{u} \circ \Gamma_\varepsilon)^2 - \frac{1}{\alpha - 1} \frac{1}{(u \circ \Gamma_\varepsilon)^{\alpha-1}} - p(t \circ \Gamma_\varepsilon) u \circ \Gamma_\varepsilon] t_0'(s) ds$$

$$= \int_0^{2\pi} \left[\left(h_0(s) + \frac{1}{\alpha - 1} \frac{1}{\varepsilon^{\alpha-1}} \right) - \frac{1}{\alpha - 1} \frac{1}{\varepsilon^{\alpha-1}} - p(t_0(s))\varepsilon \right] t_0'(s) ds$$

$$= \int_0^{2\pi} \left[h_0(s) - \varepsilon p(t_0(s)) \right] t_0'(s) ds.$$

The other integral has an analogous treatment, hence

$$\int_0^{2\pi} E \circ \bar{\Gamma}_\varepsilon \ d(t \circ \bar{\Gamma}_\varepsilon) = \int_0^{2\pi} [\mathcal{H}(t_0(s), h_0(s), \varepsilon) - \varepsilon p(\tau(s))] \tau'(s) ds,$$

where $\tau(s)$ denotes $\tau(t_0(s), h_0(s), \varepsilon)$.

Therefore, we have deduced that

$$\int_0^{2\pi} \left[h_0(s) - \varepsilon p(t_0(s)) \right] t_0'(s) ds = \int_0^{2\pi} [\mathcal{H}(t_0(s), h_0(s), \varepsilon) - \varepsilon p(\tau(s))] \tau'(s) ds.$$

Let $\varepsilon \to 0$. Then it follows that

$$\int_0^{2\pi} h_0(s)t_0'(s)ds = \int_0^{2\pi} \mathcal{H}(t_0(s), h_0(s), 0)\ \tau'(t_0(s), h_0(s), 0)ds,$$

$$\Rightarrow \int_0^{2\pi} h_0(s)t_0'(s)ds = \int_0^{2\pi} h_1(t_0(s), h_0(s))\ \frac{d}{ds}t_1(t_0(s), h_0(s))ds.$$

As $\gamma(s) = (t_0(s), h_0(s))$ we conclude

$$\int_0^{2\pi} h_0 \circ \gamma(s)d(t_0 \circ \gamma) = \int_0^{2\pi} h_0 \circ \mathcal{P} \circ \gamma(s)d(t_0 \circ \mathcal{P} \circ \gamma).$$

Hence

$$\int_\gamma h_0\ dt_0 = \int_{\mathcal{P}\circ\gamma} h_0\ dt_0.$$

□

4 Existence of periodic solutions

Theorem 4.1. *Suppose p is 2π-periodic and of class C^1. Given $N \in \mathbb{N}$, equation (1) has at least two bouncing solutions of period $2N\pi$ and having exactly one collision in the interval $[0, 2N\pi[$.*

Proof. It is enough to see the existence of two solutions of the system

$$\mathcal{P}(t_0, h_0) = (t_0 + 2N\pi, h_0), \quad t_0 \in [0, 2\pi[, \quad (t_0, h_0) \in D,$$

for each $N \geqslant 1$.

As $\mathcal{P}$ is exact symplectic by the previous section, we want to make use of the Poincaré-Birkhoff Theorem. With the notation used in the appendix, we define $\theta = t_0$, $r = h_0$ and

$$\Omega = \{(t_0, h_0) \in \mathbb{R}^2 : a < h_0 < \psi(t_0)\}$$

with $a < -\left(\frac{\alpha}{\alpha-1}\right) \|p\|_\infty^{\frac{\alpha-1}{\alpha}}$. The constant a is chosen so that

$$t_1(t_0, a) - t_0 < 2N\pi.$$

The results on comparison of solutions and Lemma 2.7 allow us to choose an a such as this one.

In order to meet the conditions of the theorem it remains to prove that for each $t_0 \in \mathbb{R}$ there exists $h_{t_0} \in]a, \psi(t_0)[$ such that

$$t_1(t_0, h_{t_0}) - t_0 > 2N\pi.$$

i) If $\psi(t_0) < +\infty$, we take an increasing sequence $\{h_{0n}\}$ converging to $\psi(t_0)$. Define

$$t_{1n} := t_1(t_0, h_{0n}).$$

Let us argue by contradiction. Suppose that

$$t_{1n} \leqslant t_0 + 2N\pi, \quad \forall\, n \in \mathbb{N}.$$

As $t_{1n} \in [t_0, t_0 + 2N\pi]$, there exists a convergent subsequence denoted by $\{t_{1k}\}$. Let $t_{1k} \to \eta \in [t_0, t_0 + 2N\pi]$.

Comparing the classical solution of equation (1) with the solution of the autonomous equation with $P = -\|p\|_\infty$, both having a collision at t_0, we find that

$$t_{1k} - t_0 \geqslant \tau(h_{0k}, -\|p\|_\infty).$$

Then letting $k \to +\infty$ we get

$$\eta \geqslant t_0 + \tau(\psi(t_0), -\|p\|_\infty),$$

and consequently

$$t_0 < t_0 + \tau(\psi(t_0), -\|p\|_\infty) \leqslant \eta \leqslant t_0 + 2N\pi.$$

Consider now $\eta^* = \frac{1}{2}(t_0 + \eta)$, the midpoint between t_0 and η. For all sufficiently large k, we have $t_0 < \eta^* < t_{1k}$. Recall the set $\mathcal{D}$ defined in Lemma 3.2. By definition of $\mathcal{D}$, $(\eta^*; t_0, h_{0k}) \in \mathcal{D}$ for all sufficiently large k. Also, as $t_1(t_0, \psi(t_0)) = +\infty$, $(\eta^*; t_0, \psi(t_0)) \in \mathcal{D}$. Therefore, letting $k \to +\infty$,

$$(\ u(\eta^*; t_0, h_{0k}), \dot{u}(\eta^*; t_0, h_{0k}) \ ) \to (\ u(\eta^*; t_0, \psi(t_0)), \dot{u}(\eta^*; t_0, \psi(t_0)) \ )$$

because this map is continuous by Lemma 3.3.

Notice that the solution $u(t; t_0, \psi(t_0))$ has no collisions after t_0. Therefore it is well defined and positive on the interval $[\eta^*, t_0 + 2N\pi + 1]$.

The theorem of continuous dependence applied to equation (1) guarantees that solution $u(t; t_0, h_{0k})$ is well defined and positive on $[\eta^*, t_0 + 2N\pi + 1]$, for all sufficiently large k.

This is a contradiction because $t_{1k} \to \eta \in [\eta^*, t_0 + 2N\pi + 1]$. Therefore it must exist $n_0 \in \mathbb{N}$ such that $t_{n_0} > t_0 + 2N\pi$, with $t_{n_0} = t_1(t_0, h_{n_0})$.

ii) If $\psi(t_0) = +\infty$, as

$$\lim_{h_0 \to +\infty} \tau(h_0, P) = +\infty,$$

whenever $P < 0$, we can choose h_0^* large enough such that

$$\tau(h_0^*, -\|p\|_\infty) > 2N\pi.$$

Then by comparison of solutions

$$t_1(t_0, h_0^*) - t_0 \geqslant \tau(h_0^*, -\|p\|_\infty) > 2N\pi.$$

We have met the hypothesis of Poincaré-Birkhoff Theorem and now we can extract the corresponding conclusions. We deduce there exist at least two solutions of the system

$$\mathcal{P}(t_0, h_0) = (t_0 + 2N\pi, h_0), \quad t_0 \in [0, 2\pi[, \quad (t_0, h_0) \in D,$$

for each $N \geqslant 1$. Setting $N = 1$, we get the existence of a 2π-periodic bouncing solution of (1). □

4.1 Future work

In future work we intend to prove the following conjecture:

Conjecture 4.1. Suppose p is 2π-periodic, of class C^1 and strictly negative. Given $N, n \in \mathbb{N}$, equation (1) has at least two bouncing solutions of period $2N\pi$ and having exactly n collisions in the interval $[0, 2N\pi[$.

This conjecture is suggested by the fact that if p is strictly negative then, as p is periodic, it has a negative maximum. Suppose that $P < 0$ is a number such that $p(t) < P$, $\forall\, t \in \mathbb{R}$. Then comparing equation (1) with

$$\ddot{u} = -\frac{1}{u^{\alpha}} + P$$

we can prove that $t_1(t_0, h_0)$ is finite for every pair $(t_0, h_0) \in \mathbb{R}^2$. Therefore we can iterate the successor map as many times as we want. In other words

$$\mathcal{P}^n = \underbrace{\mathcal{P} \circ \cdots \circ \mathcal{P}}_{\text{n times}}$$

is well defined. So if we are able to prove that $\mathcal{P}^n$ satisfies the hypothesis of Poincaré-Birkhoff Theorem then we may be able to find solutions of the system

$$\mathcal{P}^n(t_0, h_0) = (t_0 + 2N\pi, h_0), \quad t_0 \in [0, 2\pi[, \quad (t_0, h_0) \in \mathbb{R}^2,$$

which would prove the conjecture.

5 Appendix

5.1 Calculus

In this subsection we present some definitions and some technical results from calculus that we use in the proofs. The reference for the next theorem is [6].

Theorem 5.1. *(Mean value theorem for integrals) Suppose that $f : [a,b] \longrightarrow \mathbb{R}$ is continuous and that $g : [a,b] \longrightarrow \mathbb{R}$ is integrable and non-negative on $[a,b]$. Then*

$$\int_a^b f(x)g(x)\ dx = f(\xi)\int_a^b g(x)\ dx$$

for some $\xi \in [a,b]$.

In section 2, we make use of "big-O" notation. Next we make precise what we mean.

Definition 5.1. Let f, g and h be functions defined in $]x_0, x_0+\varepsilon[$ for some $\varepsilon > 0$ and $g(x), h(x) > 0$, $\forall\ x \in]x_0, x_0+\varepsilon[$.

We say $f = O(g)$ in a right neighbourhood of x_0 if there is $K > 0$ such that

$$\frac{|f(x)|}{g(x)} \leqslant K, \quad \forall x \in]x_0, x_0+\varepsilon[.$$

If every function $f = O(g)$ is such that $f = O(h)$ we write

$$O(g) \subseteq O(h).$$

An analogous definition can be given for a left neighbourhood.

Proposition 5.2. *Let $a, b, C \in \mathbb{R}$ and $C > 0$.*

1. $[C + O((x-x_0)^a)]^b \subseteq C^b + O((x-x_0)^a)$.
2. $\int_{t_0}^t O((s-t_0)^a)\ ds \subseteq (t-t_0)O((t-t_0)^a)$.

Proof. 1. Let us write the Taylor expansion around 0

$$(C+y)^b = C^b + C_1 y + C_2 y^2 + R y^3,$$

where $C_1, C_2 \in \mathbb{R}$ are the Taylor coefficients and Ry^3 is the Lagrange remainder. For values sufficiently close to 0 it is true that

$$\begin{aligned}[C + O((x-x_0)^a)]^b &= C^b + C_1 O((x-x_0)^a) + C_2 O((x-x_0)^a)^2 + RO((x-x_0)^a)^3 \\ &= C^b + O((x-x_0)^a) + O((x-x_0)^a)^2 + O((x-x_0)^a)^3\end{aligned}$$

The last line is explained by the fact that the constants are absorbed and the function R is bounded in a neighbourhood of x_0. It is easy to prove that

$$O((x-x_0)^a)^n \subseteq O((x-x_0)^{na}).$$

Hence

$$[C + O((x - x_0)^a)]^b \subseteq C^b + O((x - x_0)^a) + O((x - x_0)^{2a}) + O((x - x_0)^{3a}).$$

By definition if a function f is such that $f = O((x - x_0)^{na})$ in a neighbourhood of x_0 then there exist $\varepsilon, K > 0$ such that

$$\frac{|f(x)|}{(x - x_0)^{na}} \leqslant K, \quad \forall\, x \in]x_0, x_0 + \varepsilon[.$$

Therefore

$$\frac{|f(x)|}{(x - x_0)^a} \leqslant K(x - x_0)^{(n-1)a} \leqslant K\varepsilon^{(n-1)a}.$$

Thus $f = O((x - x_0)^a)$. That is $O((x - x_0)^{na}) \subseteq O((x - x_0)^a)$.

Consequently,

$$[C + O((x - x_0)^a)]^b \subseteq C^b + O((x - x_0)^a).$$

□

2. Let us prove that

$$\frac{1}{t - t_0} \int_{t_0}^{t} O((s - t_0)^a)\ ds \subseteq O((t - t_0)^a).$$

Let f be a function such that $f = O((s - t_0)^a)$. Then, by definition, there exists $\varepsilon > 0$ and $K > 0$ such that

$$\left| \frac{1}{t - t_0} \int_{t_0}^{t} f(s)\ ds \right| \leqslant \frac{1}{t - t_0} \int_{t_0}^{t} K(s - t_0)^a\ ds, \quad \forall t \in]t_0, t_0 + \varepsilon[,$$

for some $K, \varepsilon > 0$. Also,

$$\frac{1}{t - t_0} \int_{t_0}^{t} K(s - t_0)^a\ ds = \frac{K}{a + 1}(t - t_0)^a.$$

Therefore,

$$\frac{\left| \frac{1}{t - t_0} \int_{t_0}^{t} f(s)\ ds \right|}{(t - t_0)^a} \leqslant \frac{K}{a + 1}, \quad \forall t \in]t_0, t_0 + \varepsilon[.$$

Now this is enough to prove what we want because

$$\int_{t_0}^{t} f(s)\ ds = (t - t_0) \frac{1}{t - t_0} \int_{t_0}^{t} f(s)\ ds = (t - t_0) O((t - t_0)^a).$$

□

Proposition 5.3. *Let us consider a continuous function $f : I \subset \mathbb{R} \longrightarrow \mathbb{R}$, a class of continuous functions $f^\varepsilon : I^\varepsilon \subset \mathbb{R} \longrightarrow \mathbb{R}$ defined for $\varepsilon > 0$ in a neighbourhood of 0. Suppose that $[a,b] \subset I \cap I^\varepsilon$ for sufficiently small ε and also that*

$$f^\varepsilon(t) \longrightarrow f(t)$$

uniformly in $t \in [a,b]$.

1. *If $h : \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function then*

$$h(f^\varepsilon(t)) \longrightarrow h(f(t))$$

   uniformly in $t \in [a,b]$.

2. *Let h^ε be the inverse of f^ε and h the inverse of f on $[a,b]$. If h is continuous then*

$$h^\varepsilon(s) \longrightarrow h(s)$$

   uniformly in $s \in [f(a), f(b)]$.

Proof. 1. Notice that, since f^ε converges uniformly to f in $[a,b]$, then f^ε is uniformly bounded in this interval, i.e., there are $M, r > 0$ such that

$$|f^\varepsilon(t)| < M$$

for all $t \in [a,b]$ and $\varepsilon < r$. Fix $\mu > 0$. By the uniform continuity of h in the interval $[-M, M]$, there is $\delta > 0$ such that

$$\forall\, x, y \in [-M, M],\ |x - y| < \delta \ \Rightarrow\ |h(x) - h(y)| < \mu.$$

Now we know that there exists $r' > 0$ such that

$$|f^\varepsilon(t) - f(t)| < \delta$$

for all $t \in [a,b]$ and $\varepsilon < r'$. Letting $r_\delta = \min\{r, r'\}$ then

$$|h(f^\varepsilon(t)) - h(f(t))| < \mu$$

for all $t \in [a,b]$ and $\varepsilon < r_\delta$. □

2. If $s \in [f(a), f(b)]$ then

$$|h^\varepsilon(s) - h(s)| = |h^\varepsilon(f^\varepsilon(t)) - h(f^\varepsilon(t))|$$
$$\leqslant |h^\varepsilon(f^\varepsilon(t)) - h(f(t))| + |h(f(t)) - h(f^\varepsilon(t))| = |h(f(t)) - h(f^\varepsilon(t))|.$$

As h is a continuous function and f^ε converges uniformly to f in $[a,b]$, by the item 1. of Proposition 5.3., the function $h \circ f^\varepsilon$ converges uniformly to $h \circ f$ on the interval $[a,b]$. From the inequality, we deduce that h^ε converges uniformly to h. □

Next we give the definition of class $C^{1,0}$ which we use in chapter 4.

Definition 5.2. Given $V \subseteq \mathbb{R}^n$ and a function $f : V \times [0, \varepsilon^*] \longrightarrow \mathbb{R}$, where $f = f(x_1, \cdots, x_n; \varepsilon)$, we say that f is of class $C^{1,0}$ if it is continuous, the map

$$(x_1, \cdots, x_n) \mapsto f(x_1, \cdots, x_n; \varepsilon)$$

is of class C^1 for each $\varepsilon \in [0, \varepsilon^*]$ and the partial derivatives

$$\frac{\partial f}{\partial x_1}(x_1, \cdots, x_n; \varepsilon), \quad \cdots \quad , \frac{\partial f}{\partial x_n}(x_1, \cdots, x_n; \varepsilon)$$

are continuous on all variables.

Next lemma is a modified version of the implicit function theorem and it is used in chapter 4.

Lemma 5.4. *Given two intervals I and J, an open set $V \subseteq \mathbb{R}^n$ and a function $F : I \times V \times J \longrightarrow \mathbb{R}$ of class $C^{1,0}$, let $(s_0, z_0; \mu_0) \in I \times V \times J$ be such that $F(s_0, z_0; \mu_0) = 0$ and $\frac{dF}{ds}(s_0, z_0; \mu_0) \neq 0$. Then there is an open interval $I_1 \times V_1 \times J_1 \subseteq I \times V \times J$ containing $(s_0, z_0; \mu_0)$ and a function of class $C^{1,0}$*

$$f : V_1 \times J_1 \longrightarrow \mathbb{R}, \quad s = f(z; \mu)$$

satisfying $f(z_0; \mu_0) = s_0$ and such that

$$[(s, z; \mu) \in I_1 \times V_1 \times J_1 \textit{ and } F(s, z; \mu) = 0] \Leftrightarrow s = f(z; \mu).$$

5.2 Integration of differential forms

Given two manifolds M and N, a smooth map $F : M \longrightarrow N$ and a 1-form ω on N, we define a 1-form on M, denoted by $F^*\omega$, called the *pullback of ω by F*.

It is computed vary easily. Suppose u is a continuous real-valued function on N. Then

$$F^*(u\omega) = (u \circ F)F^*\omega.$$

If in addition u is smooth, then

$$F^*du = d(u \circ F).$$

A 1-form ω on M is said to be *exact* on M if there is a smooth function f defined on M such that $\omega = df$. The 1-form ω is said to be *conservative* if the line integral of ω over every piecewise smooth closed curve segment is zero.

Theorem 5.5. *A smooth 1-form on M is conservative if and only if it is exact.*

Theorem 5.6. *If $F : M \longrightarrow N$ is a diffeomorphism, then*

$$\int_N \omega = \int_M F^*\omega$$

if F is orientation-preserving and

$$\int_N \omega = -\int_M F^*\omega$$

if F is orientation-reversing.

Theorem 5.7. *If $F : M \longrightarrow N$ is a local diffeomorphism and N has an orientation then M has a unique orientation, called the* ***pullback orientation induced by F****, such that F is orientation-preserving.*

To know more about these subjects see [9].

5.3 Differential Equations

This section is inspired by the beautiful class notes by Rafael Ortega in [2].

A differential equation is first and foremost a statement that relates a function values with those of its derivatives. In this text we will always reduce differential equations to first order systems, i. e., equations containing only first derivatives. Moreover we are only interested in Ordinary Differential Equations.

Now, throughout this text I want to establish the following analogy. A differential equation will be a statement describing a particle's path through a velocity field so that to a differential equations is always associated a vector field. Let $D \subseteq \mathbb{R} \times \mathbb{R}^d$ be an *open* and *connected* set. We will denote the points of D as (t, x) with $t \in \mathbb{R}$ and $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$. Let's consider a *vector field* defined in D

$$X : D \longrightarrow \mathbb{R}^d, \quad (t, x) \longmapsto X(t, x) = (X_1(t, x), \ldots, X_d(t, x))$$

and we will always let X be a *continuous* map.

To each vector field we can associate the differential equation

$$\dot{x} = X(t, x),$$

where $\dot{x}$ denotes differentiation with respect to t. Given an interval I in $\mathbb{R}$, we say that a function $x : I \to \mathbb{R}^d$ is a *solution* of the differential equation if:

(i) $x(t)$ is differentiable in I;

(ii) $(t, x(t)) \in D$ for each $t \in I$;

(iii) $\dot{x}(t) = X\,(t, x(t))$, $\forall\, t \in I$.

We will need some standard results in the classical theory of differential equations as we go along. For the sake of completion I'll state them next to keep them in mind.

Given $(t_0, x_0) \in D$ we call *Initial Value Problem* or *Cauchy problem* to

$$\dot{x} = X(t, x), \quad x(t_0) = x_0$$

Theorem 5.8. *(Cauchy-Peano) If the vector field X is continuous on D and $(t_0, x_0) \in D$ then there exists a solution for the initial value problem defined on some interval I containing t_0 in its interior.*

We say there is *uniqueness* for the Cauchy problem if $x_1(t) = x_2(t) \ \ \forall t \in I_1 \cap I_2$ and for any $x_i : I_i \longrightarrow \mathbb{R}^d$, $i = 1, 2$, which are solutions.

Theorem 5.9. *(Uniqueness) Suppose there exists the Jacobian $\frac{\partial X}{\partial x}(t, x)$ on all points $(t, x) \in D$. Furthermore, suppose the function*

$$(t, x) \in D \mapsto \frac{\partial X}{\partial x}(t, x) \in \mathbb{R}^{d \times d}$$

is continuous. Then there is uniqueness for the Cauchy problem with any initial condition $(t_0, x_0) \in D$.

We say that $X : D \longrightarrow \mathbb{R}^d$ is *locally Lipschitz* in x if, for each compact set $K \subset D$, there exists $L > 0$ such that

$$\|X(t, x) - X(t, y)\| \leqslant L\|x - y\|$$

for all (t, x), $(t, y) \in K$.

Theorem 5.10. *(Picard-Lindelöf Theorem) If the vector field X is continuous on D and locally Lipschitz in x on an open set $V \subseteq D$ then for each initial condition $(t_0, x_0) \in V$ there is a unique solution for the Cauchy problem.*

Proposition 5.11. *(Existence of Maximal Solution) Assume there is uniqueness for the Cauchy Problem for each initial condition in D. Then there exists one maximal solution for the Cauchy Problem.*

Theorem 5.12. *(Extremes of Maximal Intervals) Suppose $x(t)$ is a maximal solution defined on $]t_0, t_1[$ and that $t_0 > -\infty$.*

Then at least one of the following holds:

(i) $\lim\limits_{t \to t_0^+} \|x(t)\| = +\infty$.

(ii) There exists a decreasing sequence $(t_n)_{n \in \mathbb{N}}$ to t_0 such that $x(t_n) \to \xi$ where $(t_0, \xi) \in \partial D$, i.e., (t_0, ξ) is in the border of D.

Theorem 5.13. *(Continuous Dependence on Initial Conditions) Suppose the vector field X is continuous and that there is uniqueness for the Cauchy Problem for every initial condition in D. Let us fix an initial condition $(t_0, x_0) \in D$ and suppose that the solution $x(t)$ of*

$$\dot{x} = X(t, x), \quad x(t_0) = x_0$$

is defined in $[a, b]$ with $a < t_0 < b$.

Let $(t_{0n}, x_{0n}) \in D$ be a sequence of initial conditions that satisfy

$$t_{0n} \to t_0, \quad x_{0n} \to x_0.$$

Therefore there exists an $N \in \mathbb{N}$ such that if $n \geqslant N$ the problem

$$\dot{x} = X(t, x), \quad x(t_{0n}) = x_{0n}$$

has a solution $x_n(t)$ defined on $[a, b]$. Moreover

$$\lim_{n\to+\infty} x_n(t) = x(t)$$

uniformly on $[a, b]$.

Theorem 5.14. *(General Solution) Suppose X is a continuous vector field for which there is uniqueness for the Cauchy problem for every initial condition $(t_0, x_0) \in D$. Denoting the solution to the problem*

$$\dot{x} = X(t, x), \; x(t_0) = x_0$$

by $x(t; t_0, x_0)$, let it be defined in the maximal interval $I_{(t_0,x_0)}$ containing t_0.

Then the function

$$x : \mathcal{D} \subseteq \mathbb{R} \times D \longrightarrow \mathbb{R}^d, \quad (t; t_0, x_0) \mapsto x(t; t_0, x_0),$$

where $\mathcal{D} = \{(t; t_0, x_0) \in \mathbb{R} \times D : \; t_0 \in I_{(t_0,x_0)}\}$, is continuous and $\mathcal{D}$ is an open set.

Theorem 5.15. *(Differentiability with respect to initial conditions) Suppose the vector field X is continuous, there exists the Jacobian $\frac{\partial X}{\partial x}(t, x)$ on all points $(t, x) \in D$ and the function*

$$(t, x) \in D \mapsto \frac{\partial X}{\partial x}(t, x) \in \mathbb{R}^{d\times d}$$

is continuous. Then the general solution

$$x : \mathcal{D} \subseteq \mathbb{R} \times D \longrightarrow \mathbb{R}^d, \quad (t; t_0, x_0) \mapsto x(t; t_0, x_0),$$

is differentiable and its derivatives are

1. $$\frac{\partial x}{\partial t}(t; t_0, x_0) = \dot{x}(t; t_0, x_0) = X(t; x(t; t_0, x_0));$$

2.

$$\frac{\partial x}{\partial t_0}(t; t_0, x_0) = y(t),$$

where $y(t)$ is the unique solution of the problem

$$\dot{y} = \frac{\partial X}{\partial x}(t; x(t; t_0, x_0))y, \quad y(t_0) = -X(t_0, x_0);$$

3.

$$\frac{\partial x}{\partial x_0}(t; t_0, x_0) = Y(t),$$

where $Y(t)$ is the unique solution of the problem

$$\dot{Y} = \frac{\partial X}{\partial x}(t; x(t; t_0, x_0))Y, \quad Y(t_0) = Id$$

5.3.1 Comparison of solutions

Next we present some useful results we used in the section with the same title.

Theorem 5.16. *Suppose the vector field X is continuous, there exists the Jacobian $\frac{\partial X}{\partial x}(t, x)$ on all points $(t, x) \in D$ and the function*

$$(t, x) \in D \mapsto \frac{\partial X}{\partial x}(t, x) \in \mathbb{R}^{d \times d}$$

is continuous. Suppose now that D is convex and that

$$\frac{\partial X_i}{\partial x_j}(t, x) \geqslant 0, \quad \forall\ i \neq j.$$

Let x_1 and x_2 be two solutions of

$$\dot{x}(t) = X(t, x)$$

defined on maximal intervals I_1 and I_2, respectively, with $t_0 \in I_1 \cap I_2$ and satisfying

$$x_1(t_0) \leqslant x_2(t_0).$$

Then

$$x_1(t) \leqslant x_2(t), \quad \forall\ t \geqslant t_0.$$

Next we introduce an order on $\mathbb{R}^n$. We say that $y \leqslant x$ if $y_i \leqslant x_i$, $\forall i \in \{1, ..., n\}$, where the subscripts denote the components of each vector.

We say that the vector field X is of type-K if for each $i \in \{1, ..., n\}$ it satisfies

$$X_i(t, a) \leqslant X_i(t, b)$$

whenever $a, b \in D$ are such that $a \leqslant b$ and $a_i = b_i$.

Theorem 5.17. *Suppose the vector field X is continuous and of type-K. Let x be a solution of*

$$\dot{x}(t) = X(t, x)$$

defined in an interval I and y a continuous function defined on I satisfying the inequality

$$\dot{y}(t) \geqslant X(t, y).$$

Let $t_0 \in I$ be such that $x(t_0) \leqslant y(t_0)$. Then

$$x(t) \leqslant y(t), \quad \forall\ t > t_0.$$

The same holds exchanging the inequalities.

To learn more, please see [7].

5.4 Poincaré-Birkhoff Theorem

On the next we follow [1].

Let us consider the domain in the plane

$$\Omega = \{(\theta, r) \in \mathbb{R}^2 : a < r < \psi(\theta)\}$$

where a is a constant and $\psi : \mathbb{R} \longrightarrow]a, +\infty]$ is a 2π periodic lower semi-continuous function. Let $\mathcal{P}$ be a injective map

$$\mathcal{P} : \overline{\Omega} \longrightarrow \mathbb{R}^2, \quad \mathcal{P}(\theta, r) = (\theta_1, r_1)$$

and define

$$\theta_1 = T(\theta, r), \quad r_1 = R(\theta, r),$$

where $T, R : \mathbb{R}^2 \longrightarrow \mathbb{R}$ are C^1 functions satisfying

$$T(\theta + 2\pi, r) = T(\theta, r) + 2\pi, \quad R(\theta + 2\pi, r) = R(\theta, r).$$

We say that $\mathcal{P}$ is *exact symplectic* if the differential form $r_1 d\theta_1 - r d\theta$ is exact in the cylinder, i.e., there is a function $G \in C^1(\overline{\Omega})$ such that

$$dG = r_1 d\theta_1 - r d\theta$$

and

$$G(\theta + 2\pi, r) = G(\theta, r)$$

for each $(\theta, r) \in \overline{\Omega}$.

We say that $\mathcal{P}$ is a *twist map* if the function $r \in]a, \psi(\theta)[\mapsto T(\theta, r)$ is strictly increasing for each $\theta \in \mathbb{R}$.

Theorem 5.18. *(Simplified Poincaré-Birkhoff Theorem) Suppose $\mathcal{P}$ is an exact symplectic twist map in the above conditions. Let us fix an integer N and assume that for each $\theta \in \mathbb{R}$, there exists $r_\theta \in]a, \psi(\theta)[$ with*

$$T(\theta, a) < \theta + 2N\pi < T(\theta, r_\theta).$$

Then the system

$$\begin{cases} T(\theta, r) = \theta + 2N\pi \\ R(\theta, r) = r, \quad \theta \in [0, 2\pi[, \ (\theta, r) \in \Omega, \end{cases}$$

has at least two solutions.

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